Trigonometric Integrals and Trigonometric Substitutions

Trig identities (e.g., $\cos^2 x = 1 - \sin^2 x$, $\tan x = \frac{\sin x}{\cos x}$, etc.) can sometimes be used to transform an integrand into a form where a u-substitution can be used to evaluate it.

Ex. Evaluate $\int \sin^3 x\, dx$.

Notice that: $$\sin^3 x = \sin x \left(\sin^2 x\right) = \sin x\,(1 - \cos^2 x).$$

$$\int \sin^3 x\, dx = \int \sin x\,(1 - \cos^2 x) dx$$

Let $u = \cos x$
$du = -\sin x\, dx$
$-du = \sin x\, dx$

$$\int \sin x\,(1 - \cos^2 x) dx = -\int (1 - u^2) du$$

$$= -\left(u - \frac{u^3}{3}\right) + C$$

$$= -\left(\cos x - \frac{\cos^3 x}{3}\right) + C$$

$$= -\cos x + \frac{\cos^3 x}{3} + C$$

$$\int \sin^3 x\, dx = -\cos x + \frac{\cos^3 x}{3} + C.$$

In general, when we have $\int \sin^m x \cos^n x\, dx$ we try to use $\sin^2 x + \cos^2 x = 1$ to put the integral into one of the following two forms:

$$\int (\sin^k x) \cos x\, dx \quad \text{or} \quad \int \sin x \left(\cos^j x\right) dx.$$

Then, either substitute $u = \sin x$ (in the first integral) or $u = \cos x$ (in the second integral) and we can always do this if either m or n is a positive odd integer.

Ex. Evaluate $\int \sin^4 x \cos^3 x \, dx$.

First, notice that:

$$\begin{aligned} \sin^4 x \cos^3 x &= \sin^4 x \, (\cos^2 x) \cos x \\ &= \sin^4 x \, (1 - \sin^2 x) \cos x \\ &= (\sin^4 x - \sin^6 x) \cos x \end{aligned}$$

$$\int \sin^4 x \cos^3 x \, dx = \int (\sin^4 x - \sin^6 x) \cos x \, dx$$

Let $u = \sin x$

$du = \cos x \, dx$

$$= \int (u^4 - u^6) \, du$$

$$= \frac{u^5}{5} - \frac{u^7}{7} + C$$

$$\int \sin^4 x \cos^3 x \, dx = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.$$

If the integrand just has even powers of $\sin x$ and even powers of $\cos x$ we can use:

$$\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x.$$

Ex. Evaluate $\int_0^{2\pi} \cos^2 x \, dx$.

$$\int_0^{2\pi} \cos^2 x \, dx = \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx$$

$$= \left(\frac{1}{2}x + \frac{1}{4}\sin 2x\right)\Big|_0^{2\pi}$$

$$= (\pi + 0) - (0 + 0)$$

$$= \pi.$$

Ex. Evaluate $\int \sin^4 x \, dx$.

$$\int \sin^4 x \, dx = \int (\sin^2 x)^2 \, dx$$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)^2 dx$$

$$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x) \, dx$$

$$\cos^2 2x = \frac{1}{2} + \frac{1}{2}\cos 4x$$

$$\int \sin^4 x \, dx = \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right) + C.$$

Similar strategies can be used for $\int \tan^m x \sec^n x \; dx$ because:

$$\frac{d}{dx}(\tan x) = \sec^2 x\,; \quad \frac{d}{dx}(\sec x) = \sec x \tan x\,; \quad \tan^2 x + 1 = \sec^2 x.$$

Ex. Evaluate $\int \tan^4 x \sec^4 x \; dx$.

Notice if we have $\int \tan^m x \sec^n x \; dx$ and n is even (and positive) we can always factor out $\sec^2 x$ and turn the rest of the even powers of $\sec^{n-2} x$ into powers of $\tan x$ by $\sec^2 x = \tan^2 x + 1$.

$$\begin{aligned}\tan^4 x \sec^4 x &= \tan^4 x\,(\sec^2 x)(\sec^2 x)\\ &= \tan^4 x\,(1 + \tan^2 x)(\sec^2 x)\\ &= (\tan^4 x + \tan^6 x)(\sec^2 x)\end{aligned}$$

$$\int \tan^4 x \sec^4 x \; dx = \int (\tan^4 x + \tan^6 x)(\sec^2 x)\, dx$$

Let $u = \tan x$
$du = \sec^2 x \, dx$

$$= \int (u^4 + u^6)\, du$$

$$= \frac{u^5}{5} + \frac{u^7}{7} + C$$

$$= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C.$$

If we have $\int \tan^m x \sec^n x \; dx$ and m is a positive odd integer and $n \geq 1$, then we can factor $(\tan x)(\sec x)$ from $\tan^m x\,(\sec^n x)$ and turn the even power of $\tan x$ into secants.

Ex. Evaluate $\int \tan^5 x \sec^4 x \; dx$.

$$\tan^5 x \sec^4 x = (\tan^4 x)(\sec^3 x)(\sec x \tan x)$$
$$= (\sec^2 x - 1)^2(\sec^3 x)(\sec x \tan x)$$
$$= (\sec^4 x - 2\sec^2 x + 1)(\sec^3 x)(\sec x \tan x)$$
$$= (\sec^7 x - 2\sec^5 x + \sec^3 x)(\sec x \tan x)$$

$$\int \tan^5 x \sec^4 x \; dx = \int (\sec^7 x - 2\sec^5 x + \sec^3 x)(\sec x \tan x)\, dx$$

Let $u = \sec x$

$du = \sec x \tan x \, dx$

$$= \int (u^7 - 2u^5 + u^3)\, du$$
$$= \frac{u^8}{8} - \frac{u^6}{3} + \frac{u^4}{4} + C$$
$$= \frac{\sec^8 x}{8} - \frac{\sec^6 x}{3} + \frac{\sec^4 x}{4} + C.$$

The approaches for the last couple of examples don't help us find $\int \tan x \, dx$ or $\int \sec x \, dx$. However, we can use the following "tricks" to help us.

Ex. Evaluate $\int \tan x \, dx$.

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$$

Let $u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$$= -\int \frac{du}{u} = -\ln|u| + C$$

$$\int \tan x \, dx = -\ln|\cos x| + C = \ln|\sec x| + C.$$

Ex. Evaluate $\int \sec x \, dx$.

Here we need to use a very non-obvious trick. We multiply $\sec x$ by one in the form:

$$\frac{\sec x+\tan x}{\sec x+\tan x}$$

$$\int \sec x \, dx = \int \sec x \left(\frac{\sec x+\tan x}{\sec x+\tan x}\right) dx$$

$$= \int \frac{\sec^2 x+\sec x \tan x}{\sec x+\tan x} dx$$

Notice that the numerator is now the derivative of the denominator.

Let $u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x)dx$

$$= \int \frac{du}{u} = \ln|u| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C.$$

Ex. Evaluate $\int \tan^3 x \, dx$.

$$\int \tan^3 x \, dx = \int \tan x \, (\tan^2 x) \, dx = \int \tan x \, (\sec^2 x - 1) \, dx$$

$$= \int \tan x \sec^2 x \, dx - \int \tan x \; dx$$

Let $u = \tan x$

$du = \sec^2 x \, dx$

$$= \int u \, du - \ln|\sec x| + C$$

$$= \frac{1}{2}u^2 - \ln|\sec x| + C$$

$$= \frac{1}{2}\tan^2 x - \ln|\sec x| + C.$$

Trigonometric Substitutions

Integrands, which include expressions of the form $\sqrt{a^2 - u^2}$, $\sqrt{u^2 - a^2}$, or $\sqrt{u^2 + a^2}$, can sometimes be evaluated using Pythagorean trig identities (e.g. $\sin^2\theta + \cos^2\theta = 1$).

Expression	Substitution	Trig Identity
$\sqrt{a^2 - u^2}$	$u = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + u^2}$	$u = a\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{u^2 - a^2}$	$u = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

As with integration by parts, one should be on the lookout to see if a simple substitution will allow us to evaluate the integral before using a trig substitution.

Ex. Evaluate $\int \frac{x}{\sqrt{x^2+9}}\, dx$.

Let $u = x^2 + 9$

$du = 2x\, dx$

$\frac{1}{2} du = x\, dx$

$$\int \frac{x}{\sqrt{x^2+9}}\, dx = \int \frac{1}{\sqrt{u}}\left(\frac{1}{2}\right)\, du$$

$$= \frac{1}{2}\int u^{-\frac{1}{2}}\, du$$

$$= u^{\frac{1}{2}} + C$$

$$= \sqrt{x^2+9} + C.$$

We could also evaluate this integral with a trig substitution, but it's more difficult.

Ex. Evaluate $\int \frac{\sqrt{x^2-5}}{x}\ dx.$

In this case, $a^2 = 5$ so $a = \sqrt{5}.$
Let $x = \sqrt{5}\sec\theta$
$dx = \sqrt{5}(\sec\theta)\tan\theta\ d\theta$

$$\int \frac{\sqrt{x^2-5}}{x}\ dx = \int \frac{\sqrt{5\sec^2\theta-5}}{\sqrt{5}\sec\theta}\left(\sqrt{5}(\sec\theta)(\tan\theta)\right)\ d\theta$$

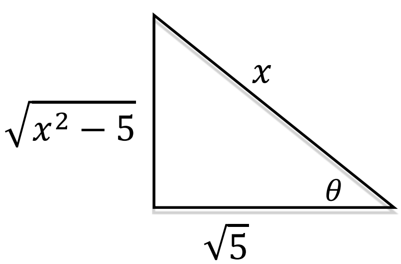
$$= \int \frac{\sqrt{5}\sqrt{\sec^2\theta-1}}{\sqrt{5}\sec\theta}\left(\sqrt{5}(\sec\theta)(\tan\theta)\right)\ d\theta$$

$$= \sqrt{5}\int \frac{\sqrt{\tan^2\theta}}{\sec\theta}\left((\sec\theta)(\tan\theta)\right) d\theta$$

$$= \sqrt{5}\int \tan^2\theta\ d\theta$$

$$= \sqrt{5}\int (\sec^2\theta - 1)\ d\theta$$

$$= \sqrt{5}(\tan\theta - \theta) + C$$



has $\frac{x}{\sqrt{5}} = \sec\theta$

so we know $\frac{\sqrt{x^2-5}}{\sqrt{5}} = \tan\theta$ and $\theta = \cos^{-1}\left(\frac{\sqrt{5}}{x}\right)$

$$\int \frac{\sqrt{x^2-5}}{x}\ dx = \sqrt{5}\left(\frac{\sqrt{x^2-5}}{\sqrt{5}} - \cos^{-1}\left(\frac{\sqrt{5}}{x}\right)\right) + C.$$

Ex. Let's revisit $\int \frac{x}{\sqrt{x^2+9}}\, dx.$

Let $x = 3\tan\theta$
$dx = 3(\sec^2\theta)d\theta$

$$\int \frac{x}{\sqrt{x^2+9}}\quad dx = \int \frac{3\tan\theta}{\sqrt{9\tan^2\theta+9}}(3\sec^2\theta)\, d\theta$$

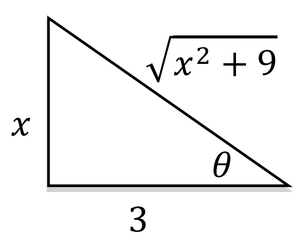
$$= 9\int \frac{\tan\theta\sec^2\theta}{3\sqrt{\tan^2\theta+1}}\, d\theta$$

$$= 3\int \frac{\tan\theta\sec^2\theta}{\sqrt{\sec^2\theta}}\, d\theta$$

$$= 3\int \frac{\tan\theta\sec^2\theta}{\sec\theta}\, d\theta$$

$$= 3\int \tan\theta\sec\theta\, d\theta$$

$$= 3\sec\theta + C$$



$$\sec\theta = \frac{\sqrt{x^2+9}}{3}$$

$$\int \frac{x}{\sqrt{x^2+9}}dx = 3\left(\frac{\sqrt{x^2+9}}{3}\right) + C$$

$$= \sqrt{x^2+9} + C.$$

Ex. Evaluate the following $\int_{\sqrt{2}}^{2} \frac{\sqrt{4-x^2}}{x^2}\,dx$.

Let $x = 2\sin\theta$; $\qquad x = \sqrt{2} \implies \theta = \frac{\pi}{4}$

$dx = 2\cos\theta\, d\theta$; $\qquad x = 2 \implies \theta = \frac{\pi}{2}$.

$$\int_{x=\sqrt{2}}^{x=2} \frac{\sqrt{4-x^2}}{x^2}\,dx = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \frac{\sqrt{4-4\sin^2\theta}}{4\sin^2\theta}(2\cos\theta)d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \frac{2\sqrt{1-\sin^2\theta}}{4\sin^2\theta}(2\cos\theta)d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \frac{\sqrt{\cos^2\theta}}{\sin^2\theta}(\cos\theta)d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \cot^2\theta\, d\theta = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} (\csc^2\theta - 1)\, d\theta$$

$$= -\cot\theta - \theta\Big|_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}}$$

$$= \left(-0 - \frac{\pi}{2}\right) - \left(-1 - \frac{\pi}{4}\right) = 1 - \frac{\pi}{4}.$$

Ex. Evaluate $\int \frac{x^3}{(4x^2+9)^{\frac{3}{2}}}\ dx.$

Notice that $(4x^2+9)^{\frac{3}{2}} = (4x^2+9)\sqrt{4x^2+9}$.
However, $\sqrt{4x^2+9}$ isn't quite in the form $\sqrt{x^2+a^2}$. We can remedy that by factoring out a 4.

$$\sqrt{4x^2+9} = \sqrt{4\left(x^2+\frac{9}{4}\right)} = 2\sqrt{x^2+\frac{9}{4}}$$

Now $a = \frac{3}{2}$

Let $x = \frac{3}{2}\tan\theta$

$dx = \frac{3}{2}(\sec^2\theta)d\theta$

$$\int \frac{x^3}{(4x^2+9)^{\frac{3}{2}}}\ dx = \int \frac{\left(\frac{3}{2}\tan\theta\right)^3}{\left(4\left(\frac{3}{2}\tan\theta\right)^2+9\right)^{\frac{3}{2}}}\left(\frac{3}{2}\sec^2\theta\right)d\theta$$

$$= \int \frac{\frac{27}{8}\tan^3\theta}{(9\tan^2\theta+9)^{\frac{3}{2}}}\left(\frac{3}{2}\sec^2\theta\right)d\theta$$

$$= \int \frac{\frac{81}{16}\tan^3\theta\sec^2\theta}{9^{\frac{3}{2}}(\tan^2\theta+1)^{\frac{3}{2}}}\ d\theta$$

$$= \int \frac{\frac{81}{16}\tan^3\theta\sec^2\theta}{27\ (\sec^3\theta)}\ d\theta$$

$$= \frac{3}{16}\int \frac{\tan^3\theta}{\sec\theta}d\theta$$

$$= \frac{3}{16}\int \frac{\sin^3\theta}{\cos^2\theta}\,d\theta$$

$$= \frac{3}{16}\int \frac{\sin^2\theta}{\cos^2\theta}(\sin\theta)d\theta$$

Let $u = \cos\theta$

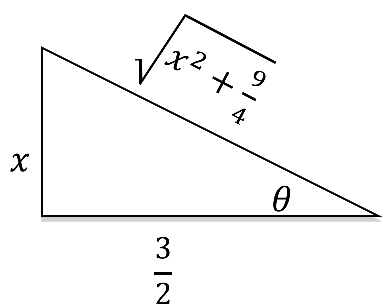
$du = -\sin\theta\, d\theta$

$$= -\frac{3}{16}\int \frac{1-u^2}{u^2}\,du$$

$$= -\frac{3}{16}\int \left(\frac{1}{u^2} - 1\right) du$$

$$= -\frac{3}{16}\left(-\frac{1}{u} - u\right) + C = \frac{3}{16}\left(\frac{1}{u} + u\right) + C$$

$$= \frac{3}{16}(\sec\theta + \cos\theta) + C.$$



$$x = \frac{3}{2}\tan\theta \;\;\Rightarrow\;\; \frac{x}{\frac{3}{2}} = \tan\theta$$

$$\sec\theta = \frac{\sqrt{x^2+\frac{9}{4}}}{\frac{3}{2}} = \frac{2}{3}\sqrt{x^2+\frac{9}{4}}$$

$$\cos\theta = \frac{\frac{3}{2}}{\sqrt{x^2+\frac{9}{4}}} = \frac{3}{2\sqrt{x^2+\frac{9}{4}}}$$

$$\int \frac{x^3}{(4x^2+9)^{\frac{3}{2}}}\,dx = \frac{3}{16}\left(\frac{2}{3}\sqrt{x^2+\frac{9}{4}} + \frac{3}{2\sqrt{x^2+\frac{9}{4}}}\right) + C.$$