Integration by Parts

Integration by parts is a method of integration that is related to the product rule for differentiation.

$$\frac{d}{dx}\big(u(x)v(x)\big) = u(x)v'(x) + v(x)u'(x)$$

$$\int \frac{d}{dx}\big(u(x)v(x)\big)\,dx = \int u(x)v'(x)\,dx + \int v(x)u'(x)\,dx$$

$$u(x)v(x) = \int u(x)v'(x)\,dx + \int v(x)u'(x)\,dx$$

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Ex. Evaluate $\int xe^x\,dx$.

Let $u = x \qquad v = e^x$

$du = dx \qquad dv = e^x dx$

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C.$$

One goal when integrating by parts is for $\int v\,du$ to be “simpler” than $\int u\,dv$.

For example, in this last problem we could have let:

$$u = e^x \qquad v = \tfrac{1}{2}x^2$$

$$du = e^x dx \qquad dv = x\,dx$$

$$\int xe^x dx = \frac{1}{2}x^2e^x - \int \frac{1}{2}x^2e^x dx$$

Thus, we started with $\int xe^x dx$ and after integrating by parts we are left with having to calculate $\frac{1}{2}\int x^2e^x dx$, which is not a “simpler” integral.

One of the biggest challenges when trying to evaluate an integral is deciding which method to use (u substitution, integration by parts, or some other method?). Unfortunately, there are no universal rules (that would work in all situations) for when to use a given method of integration. However, there are situations where one should think about using a given method of integration.

One should generally consider simpler methods of integration (e.g. a u substitution) before using a more complex method (e.g. integration by parts). Having said that here are two situations where you might want to think about using integration by parts:

1) Integrating the product of two "dissimilar" functions such as:
$$\int xe^x dx\,; \quad \int (\sin x)\, e^x dx\,; \quad \int x^2 \cos x\, dx\,; \text{ etc.}$$
2) When you have no idea what method to use:
$$\int \ln x\, dx\,; \quad \int \tan^{-1} x\, dx\,; \quad \text{etc.}$$

Ex. Evaluate $\int \tan^{-1} x\, dx$.

Let $u = \tan^{-1} x \qquad v = x$

$du = \frac{1}{1+x^2} dx \qquad dv = dx$

$$\int \tan^{-1} x\, dx = x \tan^{-1} x - \int \frac{x}{1+x^2} dx$$

To evaluate $\int \frac{x}{1+x^2} dx$, notice that the numerator is almost the derivative of the denominator (except for a factor of 2).

Let $u = 1 + x^2$

$du = 2x\, dx$

$\frac{1}{2} du = x\, dx$

$$\int \frac{x}{1+x^2} dx = \int \frac{\frac{1}{2} du}{u} = \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln|1 + x^2| + C$$
$$\int \tan^{-1} x\, dx = x \tan^{-1} x - \frac{1}{2} \ln|1 + x^2| + C.$$

Caution: Even if we have an integral that looks like it might require integration by parts, it may not.

Ex. Evaluate $\int x\, e^{x^2} dx$.

This is an integral where we are integrating the product of dissimilar functions. However, integration by parts won't work. Instead:

Let $u = x^2$

$du = 2x\, dx$

$\frac{1}{2} du = x\, dx$

$$\int x\, e^{x^2} dx = \int e^u \left(\frac{1}{2}\right) du$$

$$= \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2} + C.$$

It's not a bad idea to ask yourself if there is a simple u-substitution that will allow you to evaluate an integral before you try integrating by parts.

Ex. Evaluate the following integrals.

a) $\int x^2 \sin(x^3)\, dx$

b) $\int x^2 \sin x\, dx$

a) Let $u = x^3$

$du = 3x^2 dx$

$\frac{1}{3} du = x^2 dx$

$$\int x^2 \sin(x^3)\, dx = \int \sin(u) \left(\frac{1}{3}\right) dx$$

$$= -\frac{1}{3} \cos u + C = -\frac{1}{3} \cos(x^3) + C.$$

b) Integrate by parts:

$$u = x^2 \qquad v = -\cos x$$
$$du = 2x\,dx \qquad dv = \sin x\ dx$$

$$\int x^2 \sin x\, dx = -x^2 \cos x - \int -(\cos x)(2x)dx$$

$$= -x^2 \cos x + 2\int x \cos x\, dx$$

Integrate by parts again:

$$u = x \qquad v = \sin x$$
$$du = dx \qquad dv = \cos x\ dx$$

$$\int x \cos x\, dx = x \sin x - \int \sin x\, dx$$

$$= x \sin x + \cos x + C$$

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2\cos x + C.$$

Sometimes it's not obvious that integration by parts is giving us a simpler integral, yet we are still making progress toward an evaluation of the original integral.

Ex. Evaluate $\int e^x \cos x\, dx$.

Integrate by parts:

$$u = e^x \qquad v = \sin x$$
$$du = e^x\, dx \qquad dv = \cos x\, dx$$

$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx \qquad (*)$$

In this case, $\int e^x \sin x\, dx$ is no simpler than the original integral, $\int e^x \cos x\, dx$. However, let's integrate by parts again.

$$u = e^x \qquad v = -\cos x$$
$$du = e^x\, dx \qquad dv = \sin x\, dx$$

$$\int e^x \sin x\, dx = -e^x \cos x - \int - e^x \cos x\, dx$$
$$= -e^x \cos x + \int e^x \cos x\, dx.$$

Plugging in this new result into $(*)$ we get:

$$\int e^x \cos x\, dx = e^x \sin x - \left(-e^x \cos x + \int e^x \cos x\, dx\right)$$
$$\int e^x \cos x\, dx = e^x \sin x + e^x \cos x - \int e^x \cos x\, dx.$$

Now add $\int e^x \cos x\, dx$ to both sides of the equation:

$$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x$$
$$\int e^x \cos x\, dx = \frac{1}{2} e^x(\sin x + \cos x) + C.$$

Integration by parts can be used to evaluate definite integrals as well as indefinite integrals. The integration by parts formula for definite integrals is:

$$\boldsymbol{\int_{x=a}^{x=b} u\, dv = u(x)v(x)\Big|_{x=a}^{x=b} - \int_{x=a}^{x=b} v\, du}\,.$$

Ex. Evaluate $\int_2^e \ln x \; dx$.

Integrate by parts: $u = \ln x \qquad v = x$

$$du = \frac{1}{x}\, dx \qquad dv = dx$$

$$\int_2^e \ln x \; dx = x \ln x \Big|_{x=2}^{x=e} - \int_{x=2}^{x=e} x \left(\frac{1}{x}\right) dx$$

$$= (e \ln e - 2 \ln 2) - \int_2^e 1 \, dx = (e - 2 \ln 2) - x \Big|_{x=2}^{x=e}$$

$$= (e - 2 \ln 2) - (e - 2)$$

$$= 2 - 2 \ln 2.$$

Hyperbolic Functions

Just as we can think of a point on the unit circle, $x^2 + y^2 = 1$, as given by $(\cos(t), \sin(t))$, we can define a point on the right branch of the unit hyperbola, $x^2 - y^2 = 1; \quad x > 0$, by $(\cosh(t), \sinh(t))$, where $\cosh(t)$ is called the **hyperbolic cosine** and $\sinh(t)$ is called the **hyperbolic sine**. As with trigonometric functions we can define the other four hyperbolic functions in terms of $\cosh(t)$ and $\sinh(t)$. We define these functions as follows:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)} .$$

Ex. Find $\cosh(0)$, $\sinh(0)$, $\cosh(1)$, and $\sinh(1)$.

$$\cosh(0) = \frac{e^0+e^0}{2} = 1 \qquad \cosh(1) = \frac{e^1+e^{-1}}{2} = \frac{1}{2}\left(e+\frac{1}{e}\right)$$
$$\sinh(0) = \frac{e^0-e^0}{2} = 0 \qquad \sinh(1) = \frac{e^1-e^{-1}}{2} = \frac{1}{2}\left(e-\frac{1}{e}\right).$$

By a straight forward calculation one can check the following identities:

$$\cosh^2(x) - \sinh^2(x) = 1$$

$$\frac{d}{dx}(\sinh(x)) = \cosh(x) \qquad \int \cosh(x)\,dx = \sinh(x) + C$$
$$\frac{d}{dx}(\cosh(x)) = \sinh(x) \qquad \int \sinh(x)\,dx = \cosh(x) + C.$$

Ex. Evaluate $\int_0^1 t(sinh(t))dt$.

Let $u = t \qquad v = \cosh(t)$
$du = dt \qquad dv = \sinh(t)\,dt$

$$\int_0^1 t(sinh(t))dt = tcosh(t)|_{t=0}^{t=1} - \int_0^1 \cosh(t)\,dt$$

$$= 1(\cosh(1)) - 0(\cosh(0)) - \int_0^1 \cosh(t)\,dt$$

$$= 1\left(\frac{1}{2}\left(e+\frac{1}{e}\right)\right) - 0 - sinh(t)|_{t=0}^{t=1}$$

$$= \frac{1}{2}\left(e + \frac{1}{e}\right) - (\sinh(1) - \sinh(0))$$

$$= \frac{1}{2}\left(e + \frac{1}{e}\right) - (\frac{1}{2}\left(e - \frac{1}{e}\right) - 0)$$

$$= \frac{1}{e}\,.$$

Integration by parts can sometimes be used to find "reduction" formulas.

Ex. Show that $\int (\ln x)^n dx = x(\ln x)^n - n\int (\ln x)^{n-1} dx$.

$$u = (\ln x)^n \qquad\qquad v = x$$
$$du = n(\ln x)^{n-1}\left(\frac{1}{x}\right)\ dx \qquad\qquad dv = dx$$

$$\int (\ln x)^n dx = x\ (\ln x)^n - \int n(\ln x)^{n-1}\left(\frac{1}{x}\right) x\ dx$$

$$\int (\ln x)^n dx = x(\ln x)^n - n\int (\ln x)^{n-1} dx.$$

Ex. Use the reduction formula in the previous example to find $\int (\ln x)^2 dx$.

Putting $n = 2$ into the formula we get:

$$\int (\ln x)^n dx = x(\ln x)^n - n\int (\ln x)^{n-1} dx$$

$$\int (\ln x)^2 dx = x(\ln x)^2 - 2\int (\ln x)^1 dx. \qquad (*)$$

Now apply $n = 1$ to the formula:

$$\int (\ln x)^1 dx = x \ln x - \int (\ln x)^0 dx$$
$$= x \ln x - \int 1 dx$$
$$= x \ln x - x + C.$$

Substituting in $(*)$ we get:

$$\int (\ln x)^2 dx = x\,(\ln x)^2 - 2[x \ln x - x + C]$$
$$= x\,(\ln x)^2 - 2x \ln x + 2x + C.$$

Ex. Make a substitution and then integrate by parts to evaluate $\int \sin \sqrt{x}\, dx$.

Start by making the substitution:

$$w = \sqrt{x} = x^{\frac{1}{2}}$$
$$dw = \frac{1}{2} x^{-\frac{1}{2}} dx$$
$$dw = \frac{1}{2\sqrt{x}} dx$$

$$2\sqrt{x}\, dw = dx \quad \Rightarrow \quad 2w\, dw = dx$$

$$\int \sin \sqrt{x}\, dx = \int (\sin(w))(2w)\, dw = 2 \int w \sin w\, dw.$$

Now integrate by parts:

$$u = w \qquad v = -\cos w$$
$$du = dw \qquad dv = \sin w\, dw$$

$$\begin{aligned}\int w \sin w\, dw &= -w \cos w — \int -\cos w\, dw \\ &= -w \cos w + \int \cos w\, dw \\ &= -w \cos w + \sin w + C\end{aligned}$$

$$\int \sin \sqrt{x}\, dx = 2(-w \cos w + \sin w) + C$$

$$= -2\sqrt{x} \cos \sqrt{x} + 2 \sin \sqrt{x} + C.$$