Indeterminate Forms and L'Hospital's Rule

Def. An **indeterminate form of type $\mathbf{\frac{0}{0}}$** is written as

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

as long as the following holds:

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0.$$

An indeterminate form of type $\frac{0}{0}$ may, or may not, have a limit.

Ex. Find the limit, if it exists, of the following indeterminate forms of type $\frac{0}{0}$.

a)

$$\lim_{x\to 1}\frac{x^2-x}{x^2-1}$$

b)

$$\lim_{x\to 1}\frac{x-1}{x^2-2x+1}$$

a)

$$\lim_{x\to 1}\frac{x^2-x}{x^2-1} = \lim_{x\to 1}\frac{x(x-1)}{(x+1)(x-1)} = \lim_{x\to 1}\frac{x}{x+1} = \frac{1}{2}$$

b)

$$\lim_{x\to 1}\frac{x-1}{x^2-2x+1} = \lim_{x\to 1}\frac{x-1}{(x-1)(x-1)} = \lim_{x\to 1}\frac{1}{x-1}$$

This limit does not exist.

Def. An **indeterminate form of type $\frac{\infty}{\infty}$** is written as

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

as long as the following hold:

$$\lim_{x \to a} f(x) = \pm\infty \quad ; \quad \lim_{x \to a} g(x) = \pm\infty.$$

Again, an indeterminate form of type $\frac{\infty}{\infty}$ may, or may not, have a limit.

Ex. Find the limit, if it exists, for the following indeterminate forms of the type $\frac{\infty}{\infty}$

a)

$$\lim_{x \to \infty} \frac{x^2 + x + 2}{3x^2 + 1}$$

b)

$$\lim_{x \to \infty} \frac{x^2 + 1}{x + 1}$$

a)

$$\lim_{x \to \infty} \frac{x^2 + x + 2}{3x^2 + 1} = \lim_{x \to \infty} \frac{x^2\left(1 + \frac{1}{x} - \frac{2}{x^2}\right)}{x^2\left(3 + \frac{1}{x^2}\right)} = \lim_{x \to \infty} \frac{1 + \frac{1}{x} + \frac{2}{x^2}}{3 + \frac{1}{x^2}}$$

$$= \frac{1+0+0}{3+0} = \frac{1}{3}$$

b)

$$\lim_{x \to \infty} \frac{x^2 + 1}{x + 1} = \lim_{x \to \infty} \frac{x\left(x + \frac{1}{x}\right)}{x\left(1 + \frac{1}{x}\right)} = \lim_{x \to \infty} \frac{x + \frac{1}{x}}{1 + \frac{1}{x}} = \frac{\infty + 0}{1 + 0} = \infty$$

So there is no finite limit.

We know how to evaluate limits of the form $\lim_{x\to a}\frac{f(x)}{g(x)}$ if $f(x)$ and $g(x)$ are polynomials. However, how do we evaluate limits like $\lim_{x\to\infty}\frac{\ln x}{x}$?

L'Hospital's Rule: Suppose f and g are differentiable and $g'(x) \neq 0$ near "a" (except possibly at "a"). Suppose that:

$$\lim_{x\to a} f(x) = 0\,; \qquad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty\;; \qquad \lim_{x\to a} g(x) = \pm\infty$$

then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if the limit on the right hand side exists or is $\pm\infty$.

Note 1: Make sure the conditions of L'Hospital's Rule are satisfied before applying it. For example, if we incorrectly apply L'Hospital's Rule to:

$$\lim_{x\to 1}\frac{x+1}{x^3} = \frac{2}{1} = 2$$

then one would get the following, and incorrect, answer:

$$\lim_{x\to 1}\frac{x+1}{x^3} = \lim_{x\to 1}\frac{1}{3x^2} = \frac{1}{3}$$

Note 2: L'Hospital's Rule is also valid for one-sided limits and limits as x goes to $\pm\infty$.

Ex. Find the following limit:

$$\lim_{x\to\infty} \frac{\ln x}{x}.$$

Notice the conditions of L'Hospital's Rule are satisfied as both $f(x) = \ln x$ and $g(x) = x$ are differentiable for all $x > 0$ and:

$$\lim_{x\to\infty} \ln x = \infty\,; \qquad \lim_{x\to\infty} x = \infty$$

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{1} = 0$$

Ex. Find the following limit:

$$\lim_{x\to 0} \frac{\sin^2 x}{1-\cos x}.$$

We know:

$$\lim_{x\to 0} \sin^2 x = 0\,; \qquad \lim_{x\to 0}(1-\cos x) = 0.$$

Also the functions are both differentiable near $x = 0$.

$$\lim_{x\to 0} \frac{\sin^2 x}{1-\cos x} = \lim_{x\to 0} \frac{2\sin x\cos x}{\sin x}$$

$$= \lim_{x\to 0} 2\cos x = 2$$

Ex. Find the following limit:

$$\lim_{x\to\infty} \frac{e^x}{2x^2}$$

We know:

$$\lim_{x\to\infty} e^x = \infty\,; \qquad \lim_{x\to\infty} x^2 = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{2x^2} = \lim_{x\to\infty} \frac{e^x}{4x}$$

Notice the right hand side is still an indeterminate form of type $\frac{\infty}{\infty}$, so apply L'Hospital's Rule again:

$$\lim_{x\to\infty} \frac{e^x}{2x^2} = \lim_{x\to\infty} \frac{e^x}{4x} = \lim_{x\to\infty} \frac{e^x}{4} = \infty.$$

Ex. Find the following limit:

$$\lim_{x\to 0} \frac{\sin x - x}{x^3}$$

We know:

$$\lim_{x\to 0}(\sin x - x) = 0\,; \qquad \lim_{x\to 0} x^3 = 0$$

now apply L'Hospital's Rule:

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{\cos x - 1}{3x^2}\ .$$

The right hand side is still an indeterminate form of type $\frac{0}{0}$, so apply L'Hospital's Rule again:

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{\cos x - 1}{3x^2} = \lim_{x\to 0} \frac{-\sin x}{6x}\ .$$

Once again, the right hand side is still an indeterminate form of type $\frac{0}{0}$, so apply L'Hospital's Rule for a third time:

$$\lim_{x\to 0}\frac{\sin x - x}{x^3} = \lim_{x\to 0}\frac{\cos x - 1}{3x^2} = \lim_{x\to 0}\frac{-\sin x}{6x}$$

$$= \lim_{x\to 0}\frac{-\cos x}{6} = -\frac{1}{6}.$$

Ex. Find the following limit:

$$\lim_{x\to 0}\frac{\sin^{-1}(3x)}{x}$$

We know:

$$\lim_{x\to 0}\sin^{-1}(3x) = 0\,; \qquad \lim_{x\to 0} x = 0.$$

Now apply L'Hospital's Rule:

$$\lim_{x\to 0}\frac{\sin^{-1}(3x)}{x} = \lim_{x\to 0}\frac{\frac{3}{\sqrt{1-9x^2}}}{1} = 3\,.$$

Indeterminate Products: $0 \cdot \infty$

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \pm\infty$, then it isn't necessarily clear how to evaluate their product:

$$\lim_{x \to a}\big(f(x)g(x)\big)$$

However, we can always convert an indeterminate product of this form into either an indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

For example, we can say:

$$fg = \frac{f}{\frac{1}{g}} \qquad \text{(indeterminate form of type } \tfrac{0}{0}\text{)}$$

$$fg = \frac{g}{\frac{1}{f}} \qquad \text{(indeterminate form of type } \tfrac{\infty}{\infty}\text{)}$$

and then apply L'Hospital's Rule.

Ex. Find the following limit

$$\lim_{x \to 0^+} 2x \ln x\,.$$

We can rewrite $x \ln x$ in either of the two following ways:

$$2x \ln x = \frac{\ln x}{\frac{1}{2x}} \quad \text{or} \quad 2x \ln x = \frac{2x}{\frac{1}{\ln x}}\,.$$

Let's try the first expression.

$$\lim_{x\to 0^+} 2x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\left(\frac{1}{2x}\right)}$$

This is an indeterminate form of the type $-\frac{\infty}{\infty}$.

Now let's apply L'Hospital's Rule:

$$\lim_{x\to 0^+} 2x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\left(\frac{1}{2x}\right)} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\left(-\frac{1}{2x^2}\right)} = \lim_{x\to 0^+} (-2x) = 0$$

If we try to find this limit using the second expression we'll see that it makes the problem worse.

$$\lim_{x\to 0^+} 2x\ln x = \lim_{x\to 0^+} \frac{2x}{\left(\frac{1}{\ln x}\right)}$$

This is an indeterminate form of the type $\frac{0}{0}$.

Now apply L'Hospital's Rule:

$$\lim_{x\to 0^+} 2x\ln x = \lim_{x\to 0^+} \frac{2x}{(\ln x)^{-1}} = \lim_{x\to 0^+} \frac{2}{-(\ln x)^{-2}\left(\frac{1}{x}\right)}$$

$$= \lim_{x\to 0^+} -2x(\ln x)^2 .$$

So the second approach won't work. In general, when rewriting an indeterminate product in order to apply L'Hospital's Rule you may need to try more than one approach. Always start with the approach that looks easiest.

Ex. Find $\lim_{x\to 0^+} (\tan x)\ln x$.

$$\lim_{x\to 0^+} (\tan x)\ln x = \lim_{x\to 0^+} \frac{\ln x}{\cot x}$$

This is an indeterminate form of type $-\frac{\infty}{\infty}$.

Now let's apply L'Hospital's Rule.

$$\lim_{x\to 0^+} (\tan x)\ln x = \lim_{x\to 0^+} \frac{\ln x}{\cot x} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\csc^2 x}$$

$$= \lim_{x\to 0^+} \frac{-\sin^2 x}{x}.$$

This last expression is an indeterminate form of the type $\frac{0}{0}$, so apply L'Hospital's Rule again:

$$\lim_{x\to 0^+} (\tan x)\ln x = \lim_{x\to 0^+} \frac{-\sin^2 x}{x}$$

$$= \lim_{x\to 0^+} \frac{-2\sin x\cos x}{1} = 0.$$

Indeterminate Differences: $(\infty - \infty)$

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then we call the following an indeterminate form of type $\infty - \infty$:

$$\lim_{x\to a}[f(x) - g(x)]\,.$$

As with other indeterminate forms, this one may or may not have a limit. In this case, we want to convert the difference (through factoring, finding a common denominator, etc.) into an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Ex. Evaluate the following:

$$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right)\,.$$

Find a common denominator and subtract the fractions.

$$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right) = \lim_{x\to 0^+}\frac{(e^x - 1) - x}{x(e^x - 1)}$$

This is an indeterminate for of the type $\frac{0}{0}$, so apply L'Hospital's Rule:

$$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right) = \lim_{x\to 0^+}\frac{\frac{d}{dx}((e^x - 1) - x)}{\frac{d}{dx}(x(e^x - 1))}$$

$$= \lim_{x\to 0^+}\frac{e^x - 1}{x(e^x) + (e^x - 1)}\,.$$

This is still $\frac{0}{0}$, so apply L'Hospital's Rule again:

$$= \lim_{x \to 0^+} \left[\frac{e^x}{x(e^x) + e^x + e^x} \right]$$

$$= \frac{1}{2}$$

Indeterminate Powers: 0^0, ∞^0, and 1^∞

Several indeterminate forms can arise from:

$$\lim_{x \to a} [f(x)]^{g(x)}$$

1. Type 0°

$$\lim_{x \to a} f(x) = 0\,; \qquad \lim_{x \to a} g(x) = 0$$

2. Type ∞°

$$\lim_{x \to a} f(x) = \infty\,; \qquad \lim_{x \to a} g(x) = 0$$

3. Type 1^∞

$$\lim_{x \to a} f(x) = 1; \qquad \lim_{x \to a} g(x) = \pm\infty.$$

In each case, we let $y = [f(x)]^{g(x)}$ and take natural logarithms and then attempt to manipulate the result into a form where L'Hospital's Rule can be used.

Ex. Find the following limit: $\lim_{x\to\infty} x^{\left(\frac{1}{x}\right)}$.

This limit is type ∞^{0}.

Let $y = x^{\frac{1}{x}}$

$$\ln y = \ln x^{\frac{1}{x}} = \frac{1}{x}\ln x\,.$$

By L'Hospital's Rule: $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.$

Since $y = e^{\ln y}$, we get:

$$\lim_{x\to\infty} y = \lim_{x\to\infty} x^{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} e^{lny} = e^{0} = 1\,.$$

Ex. Calculate the following indeterminate form of the type 1^{∞}

$$\lim_{x\to 0^{+}} (1+\sin 2x)^{\frac{1}{x}}\,.$$

Let $y = (1+\sin 2x)^{\frac{1}{x}}$

$\ln y = \frac{1}{x}\ln(1+\sin(2x))$

$$\lim_{x\to 0^{+}} \ln y = \lim_{x\to 0^{+}} \frac{\ln(1+\sin(2x))}{x}$$

Now we apply L'Hospital's Rule:

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\frac{2\cos(2x)}{1+\sin(2x)}}{1} = 2$$

$$\lim_{x\to 0^+} \ln y = 2$$

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{lny} = e^2$$

$$\lim_{x\to 0^+} (1+\sin 2x)^{\frac{1}{x}} = e^2.$$