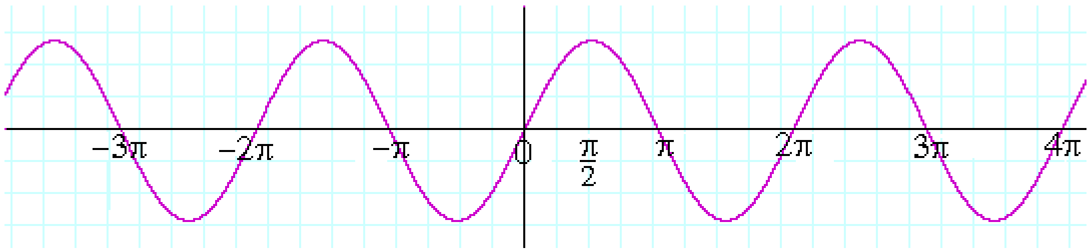
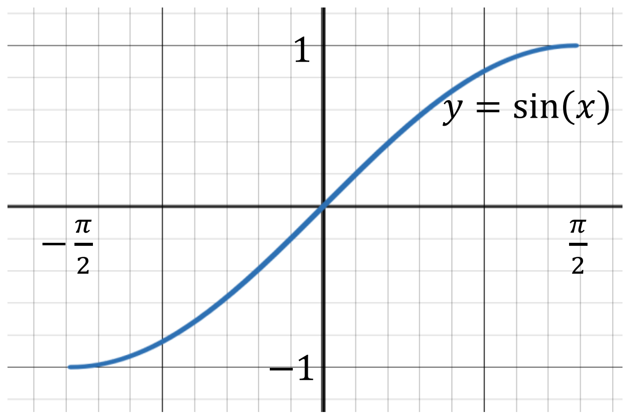
Inverse Trigonometric Functions

In order for a function to have an inverse function it must be one-to-one. We can see by the horizontal line test that $f(x) = \sin x$ is not one-to-one.



However if we restrict the domain of $y = \sin x$ to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, then it is a one-to-one function.



We call the inverse function of $f(x) = \sin x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $g(x) = \sin^{-1} x$, also called $\arcsin x$.

$$\sin^{-1} x = y \iff \sin y = x\,; \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.$$

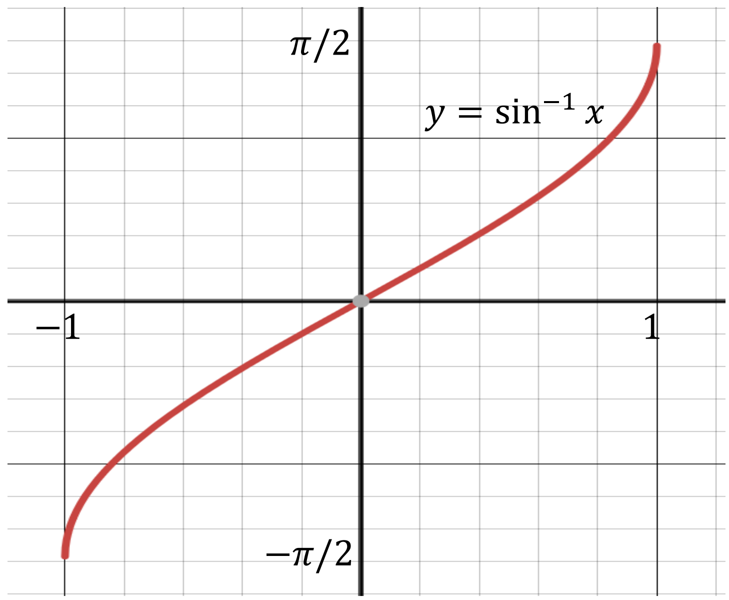
Function	Domain	Range
$f(x) = \sin x$	$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$	$-1 \leq y \leq 1$
$g(x) = \sin^{-1} x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

Since $f(x) = \sin x$ and $g(x) = \sin^{-1} x$ are inverse functions we have:

$$\sin^{-1}(\sin x) = x \text{ for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$\sin(\sin^{-1} x) = x \text{ for } -1 \leq x \leq 1.$$

Graph of $y = \sin^{-1} x$:



Ex. Evaluate the following:

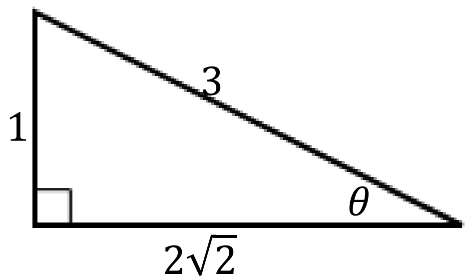
a) $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$
b) $\sec\left(\arcsin\left(\frac{1}{3}\right)\right)$

a) $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = y \quad \Leftrightarrow \quad \frac{\sqrt{2}}{2} = \sin y\,; \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

Thus, $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$, since $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $-\frac{\pi}{2} \leq \frac{\pi}{4} \leq \frac{\pi}{2}$.

b) Let $\theta = \arcsin\left(\frac{1}{3}\right) = \sin^{-1}\left(\frac{1}{3}\right)$ so $\sin\theta = \frac{1}{3}$.

Draw a right triangle where $\sin\theta = \frac{1}{3}$.



By the Pythagorean Theorem we can find the third side by:

$$1^2 + x^2 = 3^2 \quad \Rightarrow \quad x^2 = 8 \quad \Rightarrow x = 2\sqrt{2} \text{ since } x \geq 0.$$

Thus we have:

$$\sec\left(\arcsin\left(\frac{1}{3}\right)\right) = \frac{3}{2\sqrt{2}}.$$

We can find the derivative of $y = \sin^{-1} x$ using implicit differentiation.

$$\begin{aligned} y &= \sin^{-1} x \\ \sin y &= x \\ \frac{d}{dx}(\sin y) &= \frac{d}{dx}(x) \\ (\cos y)\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos y}. \end{aligned}$$

Remember that $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and $\cos y \geq 0$, so we can write:

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}\,; \qquad -1 < x < 1.$$

Similarly, if $y = \sin^{-1}(u(x))$, then by the chain rule: $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$.

Ex. Let $f(x) = \sin^{-1}(3x+1)$

a) Find the domain of $f(x)$
b) Find $f'(x)$

a) The domain for $g(x) = \sin^{-1} x$ is $-1 \le x \le 1$.
The domain for $f(x) = \sin^{-1}(3x+1)$ is $-1 \le 3x+1 \le 1$ or

$$-2 \le 3x \le 0 \quad \Rightarrow \quad -\tfrac{2}{3} \le x \le 0.$$

b)

$$f'(x) = \frac{1}{\sqrt{1-(3x+1)^2}}\frac{d}{dx}(3x+1)$$

$$= \frac{1}{\sqrt{1-(9x^2+6x+1)}}(3) = \frac{3}{\sqrt{-9x^2-6x}}.$$

Ex. Let $y = \sqrt{\arcsin(x^2-1)}$. Find $\dfrac{dy}{dx}$.

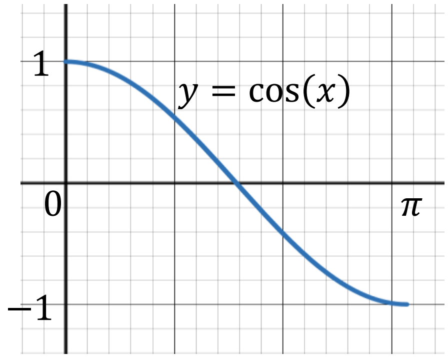
$$y = (\sin^{-1}(x^2-1))^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(\sin^{-1}(x^2-1))^{-\frac{1}{2}}\frac{d}{dx}(\sin^{-1}(x^2-1))$$

$$= \frac{1}{2\sqrt{\sin^{-1}(x^2-1)}}\frac{1}{\sqrt{1-(x^2-1)^2}}\frac{d}{dx}(x^2-1)$$

$$= \frac{1}{2\sqrt{\sin^{-1}(x^2-1)}}\frac{1}{\sqrt{1-(x^4-2x^2+1)}}(2x) = \frac{1}{2\sqrt{\sin^{-1}(x^2-1)}}\frac{2x}{\sqrt{2x^2-x^4}}.$$

For the inverse cosine, we restrict the domain of $f(x) = \cos x$ to $0 \le x \le \pi$ so that $y = \cos x$ is one-to-one.

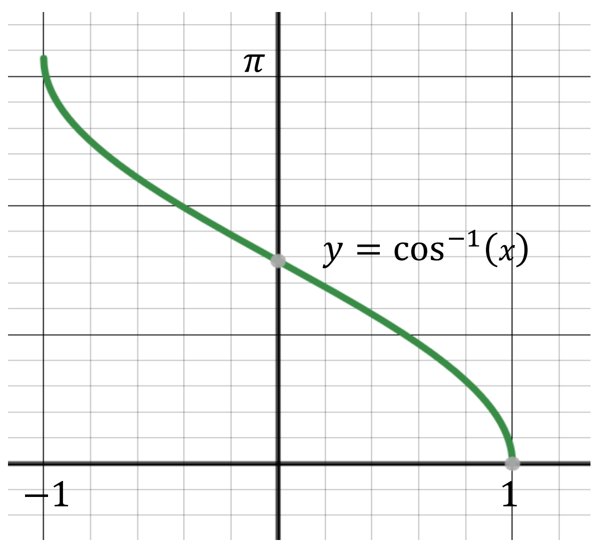


We define inverse cosine, $\cos^{-1} x$ or $\arccos x$ by:

$$\cos^{-1} x = y \iff \cos y = x; \qquad 0 \le y \le \pi$$
$$\cos^{-1}(\cos x) = x; \qquad 0 \le x \le \pi$$
$$\cos(\cos^{-1} x) = x; \qquad -1 \le x \le 1$$

Function	Domain	Range
$f(x) = \cos x$	$0 \le x \le \pi$	$-1 \le y \le 1$
$g(x) = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$

Graph of $y = \cos^{-1} x = \arccos x$:



An approach similar to the one used to find $\frac{d}{dx}(\sin^{-1} x)$ gives us:

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \; ; \qquad -1 < x < 1$$

$$\frac{d}{dx}(\cos^{-1}(u(x))) = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}.$$

Ex. Let $y = x^2 \cos^{-1}(x^3)$. Find $\frac{dy}{dx}$.

Start by using the product rule:

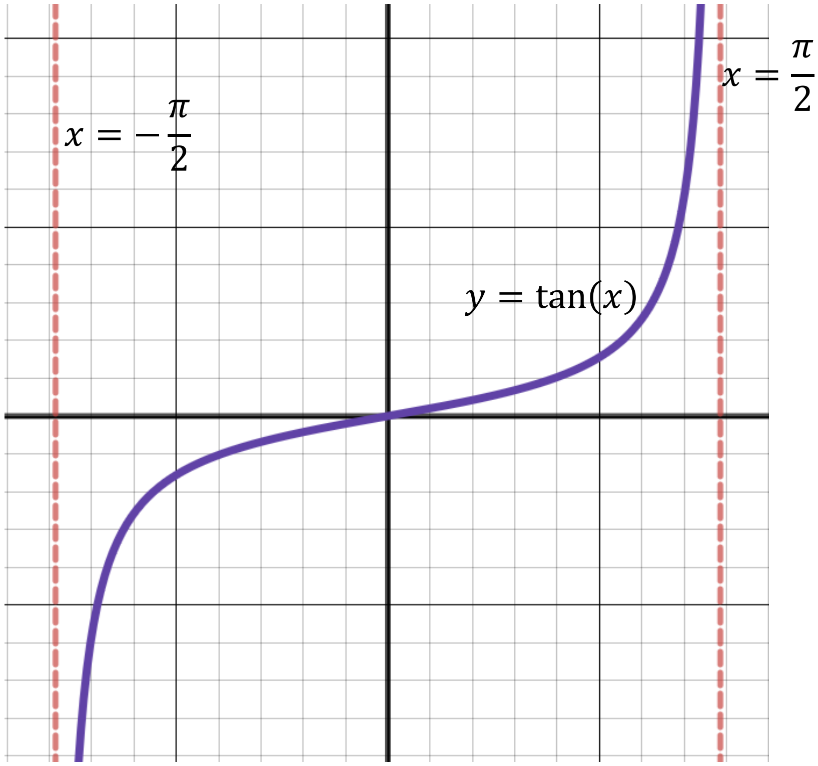
$$\frac{dy}{dx} = x^2 \frac{d}{dx}\left(\cos^{-1}(x^3)\right) + \left(\cos^{-1}(x^3)\right)\frac{d}{dx}(x^2)$$

$$= x^2\left(\frac{-1}{\sqrt{1-(x^3)^2}}\right)\frac{d}{dx}(x^3) + (2x)(\cos^{-1}(x^3))$$

$$= -x^2\left(\frac{3x^2}{\sqrt{1-x^6}}\right) + (2x)(\cos^{-1}(x^3))$$

$$= -\frac{3x^4}{\sqrt{1-x^6}} + (2x)(\cos^{-1}(x^3)).$$

$f(x) = \tan x$ is a one-to-one function on $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

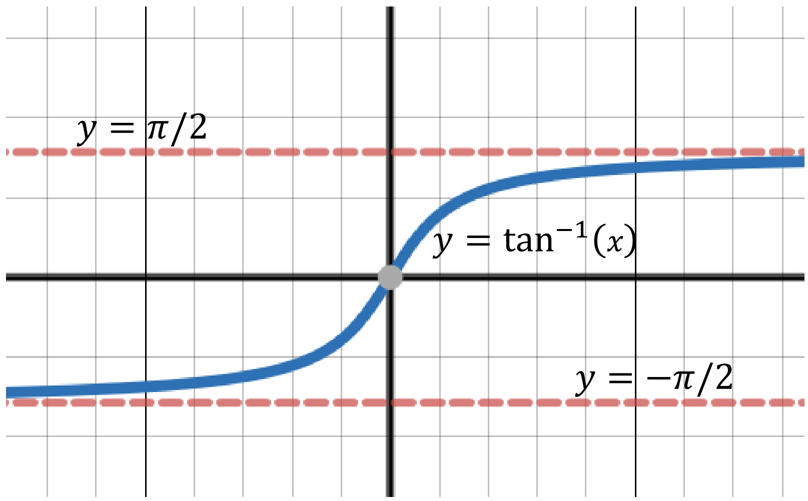


We define $\tan^{-1} x$ or $\arctan x$ by:

$$\tan^{-1} x = y \iff y = \tan x \ ; \quad -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

Function	Domain	Range
$f(x) = \tan x$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$	$-\infty < y < \infty$
$g(x) = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$.

The graph of $y = \tan^{-1} x = \arctan x$:



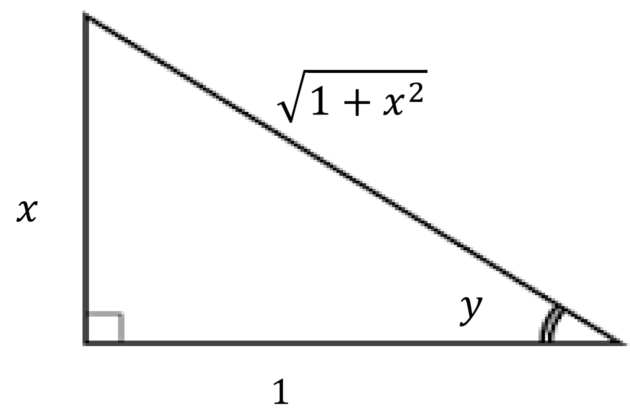
Notice:

$$\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$$

$$\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}.$$

Ex. Simplify $\csc(\tan^{-1} x)$.

Start by drawing right triangle where $\tan y = x$. Thus, by the Pythagorean theorem the hypotenuse is $\sqrt{1+x^2}$.



Now notice that $\csc(y) = \frac{\sqrt{1+x^2}}{x}$ but $\tan y = x$, thus $y = \tan^{-1} x$.

$$\csc(\tan^{-1} x) = \frac{\sqrt{1+x^2}}{x}.$$

Ex. Evaluate the following: $\lim_{x\to 0^+} \arctan(\ln x)$.

$$\lim_{x\to 0^+} \ln x = -\infty$$

$$\lim_{x\to 0^+} \tan^{-1}(\ln x) = \lim_{y\to -\infty} \tan^{-1}(y)$$

$$= -\frac{\pi}{2}.$$

To find $\frac{d}{dx}(\tan^{-1} x)$ we employ a similar approach to the one we used for finding $\frac{d}{dx}(\sin^{-1} x)$.

$$y = \tan^{-1} x$$

$$\tan y = x$$

$$\frac{d}{dx}(\tan y) = x$$

$$(\sec^2 y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$= \frac{1}{1+\tan^2 y}$$

$$= \frac{1}{1+x^2}\,.$$

So we have:

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}\,.$$

$$\frac{d}{dx}\left(\tan^{-1}\left(u(x)\right)\right) = \frac{1}{1+u^2}\frac{du}{dx}\,.$$

The inverse cosecant, secant, and cotangent are defined analogously to the other inverse trigonometric functions.

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \frac{d}{dx}(\csc^{-1} u) = -\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$$

$$\frac{d}{dx}(\cos^{-1} u) = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \frac{d}{dx}(\sec^{-1} u) = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$$

$$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2}\frac{du}{dx} \qquad \frac{d}{dx}(\cot^{-1} u) = -\frac{1}{1+u^2}\frac{du}{dx}$$

Ex. Let $y = (\tan^{-1}(x^2))^3$. Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 3(\tan^{-1}(x^2))^2\frac{d}{dx}(\tan^{-1}(x^2))$$

$$= 3(\tan^{-1}(x^2))^2\left(\frac{1}{1+(x^2)^2}\right)\left(\frac{d}{dx}(x^2)\right)$$

$$= 3(\tan^{-1}(x^2))^2\left(\frac{2x}{1+x^4}\right)$$

$$= \left(\frac{6x}{1+x^4}\right)(\tan^{-1}(x^2))^2.$$

Each derivative formula for an inverse trig function gives rise to an integral formula. The two most useful formulas are:

$$\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1} u + C$$

$$\int \frac{1}{1+u^2}\,du = \tan^{-1} u + C$$

Ex. Evaluate the following :

$$\int_0^{\frac{1}{6}} \frac{1}{\sqrt{1-9x^2}}\,dx$$

Notice that if we let $u = 3x$, then $u^2 = 9x^2$
and $du = 3\,dx$.

$$\tfrac{1}{3}\,du = dx \qquad x = 0 \;\Rightarrow\; u = 3(0) = 0$$

$$x = \tfrac{1}{6} \;\Rightarrow\; u = 3\left(\tfrac{1}{6}\right) = \tfrac{1}{2}.$$

$$\int_{x=0}^{x=\frac{1}{6}} \frac{1}{\sqrt{1-9x^2}}\,dx = \int_{u=0}^{u=\frac{1}{2}} \frac{1}{\sqrt{1-u^2}}\left(\frac{1}{3}\right)du$$

$$= \frac{1}{3}\sin^{-1} u\Big|_{u=0}^{u=\frac{1}{2}}$$

$$= \tfrac{1}{3}\left[\sin^{-1}\left(\tfrac{1}{2}\right) - \sin^{-1}(0)\right]$$

$$= \tfrac{1}{3}\left[\tfrac{\pi}{6} - 0\right] = \tfrac{\pi}{18}.$$

Ex. Evaluate the following: $\int \frac{e^x}{\sqrt{1-e^{2x}}}\, dx.$

Let $u = e^x$
$du = e^x dx$

$$\int \frac{e^x}{\sqrt{1-e^{2x}}} = \int \frac{du}{\sqrt{1-u^2}}$$

$$= \sin^{-1} u + C$$

$$= \sin^{-1}(e^x) + C$$

Ex. Evaluate: $\int \frac{1}{x^2+9}\, dx.$

$$\int \frac{1}{x^2+9}\, dx = \int \frac{1}{9\left(\frac{x^2}{9}+1\right)}\, dx = \frac{1}{9}\int \frac{1}{\left(\frac{x}{3}\right)^2+1}\, dx$$

Let $u = \frac{x}{3}$
$du = \frac{1}{3} dx$
$3du = dx$

$$= \frac{1}{9}\int \frac{1}{u^2+1}(3)\, du$$

$$= \frac{1}{3}\tan^{-1} u + C$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

Ex. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x} dx$.

Let $u = \sin x$

$du = \cos x \, dx$

When $x = 0, u = 0$ and when $x = \frac{\pi}{2}, u = 1$.

$$\int_{x=0}^{x=\frac{\pi}{2}} \frac{\cos x}{1 + \sin^2 x} dx = \int_{u=0}^{u=1} \frac{du}{1 + u^2}$$

$$= \tan^{-1} u \Big|_{u=0}^{u=1}$$

$$= \tan^{-1} 1 - \tan^{-1} 0$$

$$= \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4}.$$