Exponential Growth and Decay

If the rate of change of a quantity, $P(t)$, is proportional to the amount present (e.g. $P(t)$ could be the population of a city), then $P(t)$ satisfies the differential equation:

$$P'(t) = kP(t)$$

Def. The **relative growth rate** is written as $k = \frac{P'(t)}{P(t)}$.

The general solution to $P'(t) = k\ P(t)$ or $\frac{dP}{dt} = kP(t)$ is given by:

$$P(t) = Ce^{kt} \qquad \text{(where } C \text{ is a constant).}$$

Notice that $P'(t) = Cke^{kt}$, so:

$$P'(t) = Cke^{kt} = kP(t).$$

If $k > 0$, then $P(t)$ grows exponentially.

If $k < 0$, then $P(t)$ decays exponentially.

If we know the value of $P(t)$ at $t = 0$, then:

$$P(t) = Ce^{kt}$$

$$P(0) = Ce^{0} = C.$$

So the solution to $P'(t) = kP(t)$ is:

$$P(t) = P(0)e^{kt}.$$

Ex. A bacteria culture grows at a rate proportional to its size. At time $t = 0$, approximately 20,000 bacteria are present. In 5 hours there are 400,000 bacteria. Determine a function that expresses the size of the culture as a function of time, measured in hours. What is the relative growth rate?

$$P(0) = 20{,}000$$

$$P(t) = P(0)e^{kt} = 20{,}000e^{kt}.$$

To solve for $k =$ relative growth rate, we can use the fact that $P(5) = 400{,}000.$

$$400{,}000 = 20{,}000e^{k(5)}$$

$$20 = e^{5k} \qquad \text{take } ln \text{ of both sides}$$

$$\ln 20 = 5k$$

$$\frac{\ln 20}{5} = k$$

So the relative growth rate is $\frac{\ln 20}{5}$ and we can say:

$$P(t) = 20{,}000e^{\left(\frac{\ln 20}{5}\right)t}.$$

In this problem the relative growth rate was $\frac{\ln 20}{5} \approx .6 = 60\%/hr.$

Ex. The population of a colony of fruit flies grows according to $P(t) = P(0)e^{kt}$, where t is in days. The colony doubles in size in 9 days. Determine the value of k, the relative growth rate.

$$P(9) = 2P(0)$$

$$P(0)e^{9k} = 2P(0)$$

$$e^{9k} = 2 \qquad \text{Now take the } ln \text{ of both sides}$$

$$9k = \ln 2$$

$$k = \frac{\ln 2}{9} \approx .077.$$

The relative growth rate is approximately 7.7% per day.

Notice that to find the relative growth rate we did not need to know $P(0)$.

Radioactive decay is often described by: $P(t) = P(0)e^{kt},\ k < 0$. Frequently physicists describe the rate of decay in terms of the **half-life**, the time required for half of a given quantity to decay away.

Ex. Suppose radioactive carbon 14 has a half-life of 5,730 years.

a) A sample of carbon 14 has a mass of 250 mg. Find a formula for the mass of carbon 14 after t years.

b) Find the mass after 2,000 years to the nearest mg.

c) When will the mass be reduced to 100 mg?

a) We know that $P(t) = (250)e^{kt}$, but we need to find k. Using the half-life, 5730, we can write:

$$P(5730) = \frac{1}{2}P(0) = 125$$

$$(250)\left(e^{k(5730)}\right) = 125$$

$$e^{5730k} = \frac{1}{2}$$

$$5730k = \ln\frac{1}{2}$$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{5730} \approx -0.000121$$

Thus $\quad P(t) = (250)e^{-0.000121t}$.

b) $P(2000) = 250e^{-0.000121(2000)} \approx 196$ mg

c) $P(t) = (250)e^{-0.000121t} = 100$ mg

$$e^{-0.000121t} = \frac{100}{250} = \frac{2}{5}$$

$$-0.000121t = \ln\left(\frac{2}{5}\right)$$

$$t = -\frac{1}{0.000121}\ln\left(\frac{2}{5}\right) \approx 7{,}573 \text{ years.}$$

Newton's Law of Cooling

Newton's law of cooling says that the rate of cooling (or heating) of an object is proportional to the temperature difference between the object and its surroundings.

If we let $T(t)$ be the temperature of the object and T_s be the constant temperature of the surroundings, then Newton's law says:

$$\frac{dT}{dt} = k(T - T_s) \qquad \text{where } k \text{ is a constant.}$$

Notice that if we let $P(t) = T(t) - T_s$, then the differential equation becomes:

$$\frac{dP}{dt} = k\,P(t).$$

We know that the general solution to this differential equation is:

$$P(t) = P(0)e^{kt}.$$

$$P(t) = T(t) - T_s$$

so $$P(0) = T(0) - T_s\,.$$

Now plug into the general solution:

$$T(t) - T_s = (T(0) - T_s)e^{kt}$$

or $$T(t) = T_s + (T(0) - T_s)e^{kt}.$$

Ex. The temperature of a cup of freshly brewed coffee is $120°\text{F}$. It's left to cool in a room that is $70°\text{F}$. After 5 minutes the coffee is $100°\text{F}$.

a) What is the temperature of the coffee after another 5 minutes?

b) How long does it take the coffee to cool to $80°\text{F}$?

a) $T(t) = T_s + (T(0) - T_s)e^{kt}$ where $T_s = 70$ and $T(0) = 120$

$$T(t) = 70 + (50)e^{kt}$$

$$T(5) = 70 + 50\left(e^{k(5)}\right) = 100$$

$$50e^{5k} = 30$$

$$e^{5k} = \frac{30}{50} = \frac{3}{5}$$

$$5k = \ln\left(\frac{3}{5}\right)$$

$$k = \frac{1}{5}\ln\left(\frac{3}{5}\right) \approx -0.1022$$

$$T(t) = 70 + 50e^{-0.1022t}$$

$$T(10) = 70 + 50e^{-0,1022(10)} \approx 87.99°$$

b)

$$70 + 50e^{-0.1022t} = 80$$

$$50e^{-0.1022t} = 10$$

$$e^{-0.1022t} = \frac{1}{5}$$

$$-0.1022t = \ln\left(\frac{1}{5}\right)$$

$$t = -\frac{1}{0.1022}\ln\left(\frac{1}{5}\right)$$

≈ 15.7 minutes.

Continuously Compounded Interest

If we start with \$1,000 and a 6% interest rate compounded annually, then after 1 year we have:

$$\$1000(1+0.06) = \$1060 \quad \text{(annual compounding).}$$

After 2 years we have:

$$\big(\$1000(1+.06)\big)(1+.06) = \$1000(1.06)^2 = \$1123.60.$$

After t years we have:

$$\$1000(1.06)^t.$$

If the interest rate is compounded twice a year (i.e. semi-annually), then after 1 year we have:

$$\$1000\left(1+\frac{.06}{2}\right)^2 = \$1060.90.$$

After t years we would have:

$$\$1000\left(1+\frac{.06}{2}\right)^{2t}.$$

For a general annual interest rate of $r\%$ compounded n times per year, an initial amount of A_0 dollars will grow in t years to:

$$A(t) = A_0\left(1+\frac{r}{n}\right)^{nt}.$$

If we invest $\$1000$ at 6% annual rate for 3 years, the amount we have at the end will depend on how many times per year it is compounded.

Compounding Periods/ Year	Final Amount
1	$\$1000(1.06)^3 = \$1191.02$
2	$\$1000(1.03)^6 = \$1194.05$
4	$\$1000(1.015)^{12} = \$1195.62$
12	$\$1000(1.005)^{36} = \$1196.68$
365	$\$1000\left(1+\frac{.06}{365}\right)^{1095} = \$1197.20$

What happens if we let the number of compounding periods per year, n, go to infinity? This is called continuous compounding.

$$A(t) = \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt}$$

$$= A_0 \lim_{n\to\infty}\left[\left(1+\frac{r}{n}\right)^{\frac{n}{r}}\right]^{rt}$$

Let $m = \frac{n}{r}$

$$= A_0 \lim_{m\to\infty}\left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt}.$$

Claim: $\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m} = e.$

This limit follows from taking $f(x) = lnx$ and noticing that

$$1 = f'(1) = \lim_{h\to 0}\frac{\ln(1+h)-\ln(1)}{h} = \lim_{h\to 0}\frac{\ln(1+h)}{h} = \lim_{h\to 0^+}\frac{\ln(1+h)}{h}.$$

Now let $m = \frac{1}{h}$. As $h \to 0^+,\ m \to \infty$

$$1 = \lim_{m\to\infty}(m)ln\left(1+\frac{1}{m}\right).$$

Thus we have:

$$e = e^1 = e^{\lim_{m\to\infty}(m)ln\left(1+\frac{1}{m}\right)} = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m}.$$

Since $\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = e$ we have

$$\lim_{m\to\infty}\left[\left(1+\frac{1}{m}\right)^m\right]^{rt} = e^{rt} \text{ and thus:}$$

$$A(t) = \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt} = A_0e^{rt}.$$

Ex. If we invest \$1,000 at a 6% annual interest rate compounded continuously, then how much money will we have after 3 years?

$$A(3) = 1000e^{(.06)(3)}$$

$$= \$1{,}197.22.$$

Ex. How long does it take an investment to double if the annual interest rate is 10% compounded continuously? What is the equivalent annually compounded interest rate?

If we start with A dollars today, then compounding continuously at 10% gives us after t years:

$$A(t) = Ae^{(.1)t} = 2A$$

$$e^{.1t} = 2$$

$$.1t = \ln 2$$

$$t = \frac{1}{.1}\ln 2 = 10\ln 2$$

≈ 6.93 years to double initial investment.

For an annually compounded interest rate:

$$A(t) = A(1+r)^t = 2A\,; \qquad t = 6.93$$

$$(1+r)^{6.93} = 2$$

$$1 + r = 2^{\frac{1}{6.93}}$$

$$r = 2^{\frac{1}{6.93}} - 1$$

$$\approx\ .1052 = 10.52\%.$$

So the money will double in 6.93 years if it is compounded annually at a rate of 10.52%.