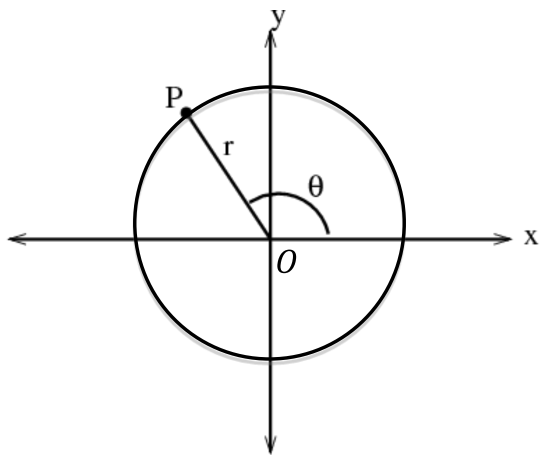
Polar Coordinates

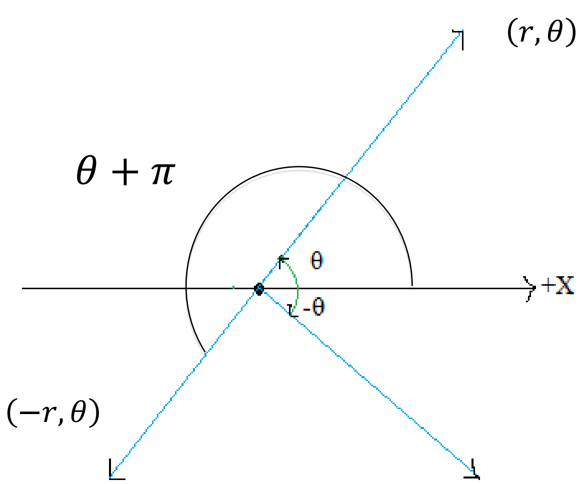
Just like Cartesian coordinates, (x, y), allow us to describe where a point is in a plane, polar coordinates also allow us to describe where a point is in a plane. In polar coordinates we start by thinking of any point P in the x- y plane lying on a circle of radius r whose center is at O, $x = 0$; $y = 0$, (in polar coordinates we call this point the **pole**). We can identify the point P by saying how far it is from the pole (r) and what angle the line segment OP makes with the x-axis (θ), measured counter-clockwise from the x-axis.



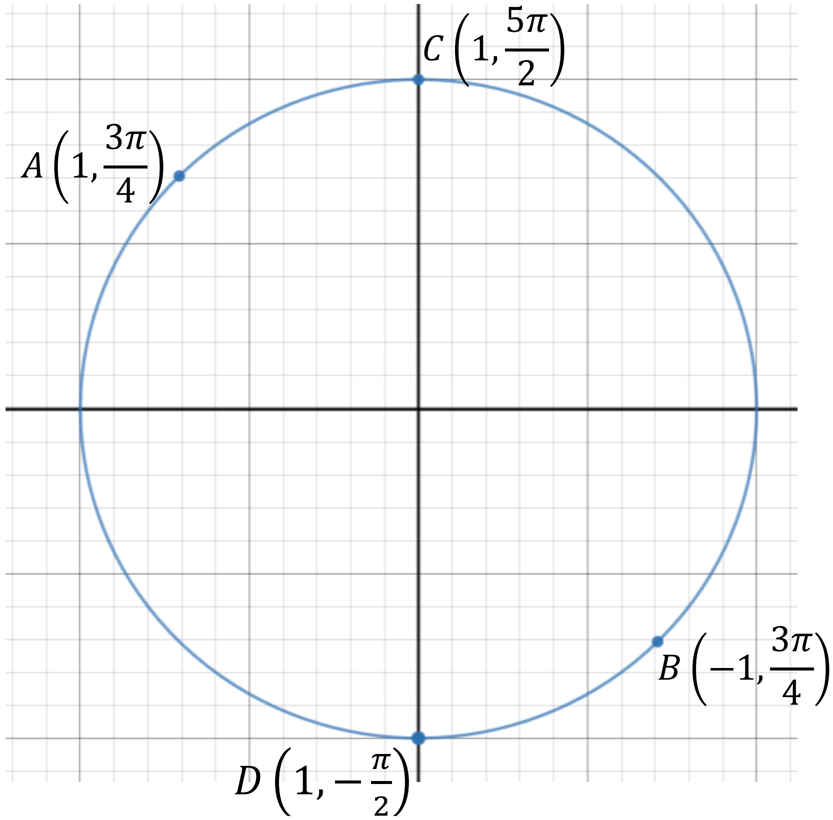
Thus, P is represented by (r, θ) in polar coordinates. Notice that $(0, \theta)$ represents the pole, O, for any value of θ. Also, if (r, θ) represents the point P, then so does $(r, \theta + 2n\pi)$ (where n is any integer).

We extend the meaning of (r, θ) for the case when $r < 0$ by saying that:

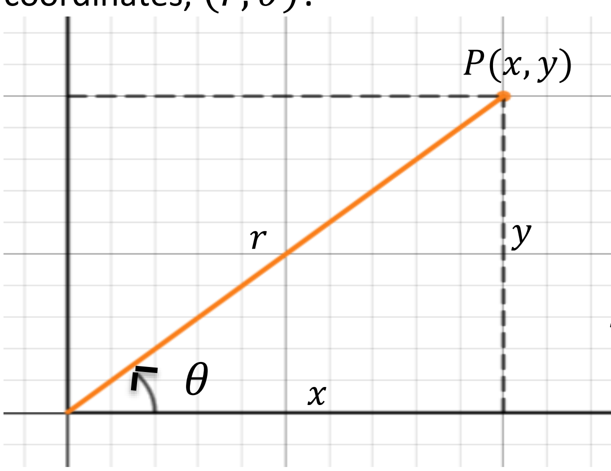
$$(-r, \theta) = (r, \theta + \pi).$$



Ex. Plot the points: $A\left(1, \frac{3\pi}{4}\right)$, $B\left(-1, \frac{3\pi}{4}\right)$, $C\left(1, \frac{5\pi}{2}\right)$, $D\left(1, -\frac{\pi}{2}\right)$.



So what is the relationship between Cartesian coordinates, (x, y), and polar coordinates, (r, θ)?



$$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$$

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin\theta = r^2$$

$$\tan\theta = \frac{y}{x}.$$

Summary of the relationships between Cartesian and Polar coordinates:

$x = r\cos\theta$	$r^2 = x^2 + y^2$
$y = r\sin\theta$	$\tan\theta = \frac{y}{x}$

Ex. Convert the points $\left(4, \frac{\pi}{6}\right)$ and $\left(3, \frac{2\pi}{3}\right)$ to Cartesian coordinates.

For $\left(4, \frac{\pi}{6}\right)$:

$$x = r\cos\theta = 4\cos\left(\frac{\pi}{6}\right) = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$$

$$y = r\sin\theta = 4\sin\left(\frac{\pi}{6}\right) = 4\left(\frac{1}{2}\right) = 2$$

$\left(4, \frac{\pi}{6}\right) = \left(2\sqrt{3}, 2\right)$ in Cartesian coordinates.

For $\left(3, \frac{2\pi}{3}\right)$:

$$x = r\cos\theta = 3\cos\left(\frac{2\pi}{3}\right) = 3\left(-\frac{1}{2}\right) = -\frac{3}{2}$$

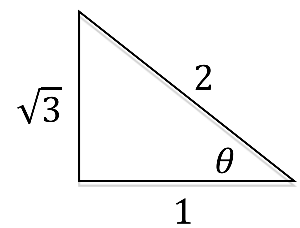
$$y = r\sin\theta = 3\sin\left(\frac{2\pi}{3}\right) = 3\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}$$

$\left(3, \frac{2\pi}{3}\right) = \left(-\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$ in Cartesian coordinates.

Ex. Represent $\left(-1, -\sqrt{3}\right)$ in polar coordinates where $r > 0$ and $0 \le \theta < 2\pi$. These conditions will guarantee a unique solution.

$$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2} = \sqrt{1+3} = 2$$

$$\tan\theta = \frac{-\sqrt{3}}{-1} = \sqrt{3}$$



$$\tan\theta = \sqrt{3} \;\Rightarrow\; \theta = \frac{\pi}{3} \text{ or } \frac{4\pi}{3}$$

Since $\left(-1, -\sqrt{3}\right)$ is in the 3rd quadrant, $\theta = \frac{4\pi}{3}$.
So $\left(-1, -\sqrt{3}\right)$ in Cartesian coordinates is $\left(2, \frac{4\pi}{3}\right)$ in polar corrdinates, as long as $r > 0$ and $0 \le \theta < 2\pi$.

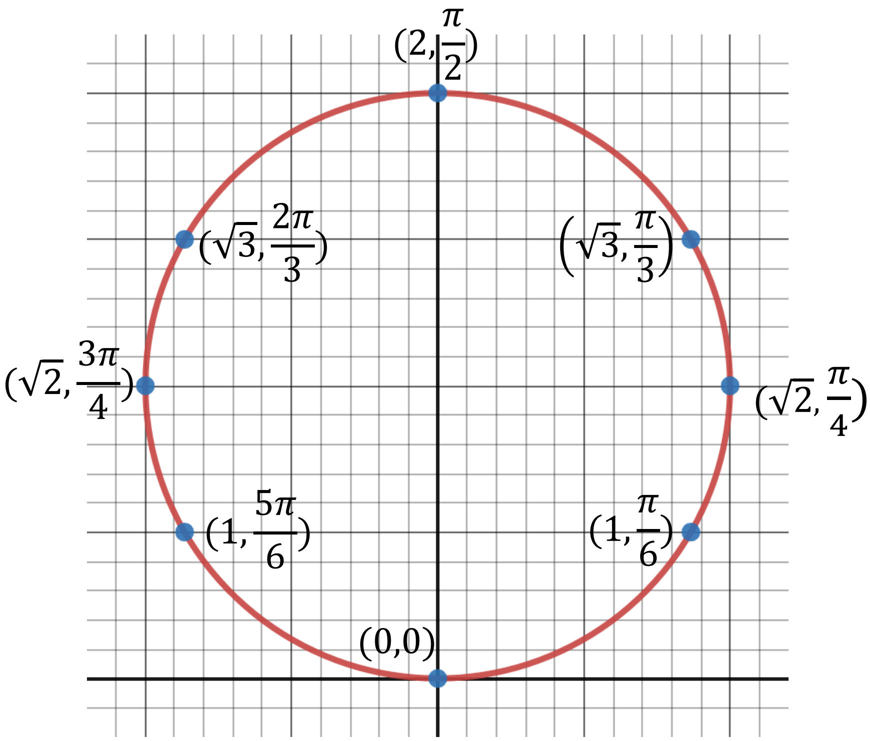
Just as we can represent curves in Cartesian coordinates by $f(x, y) = 0$, we can represent curves in polar coordinates by $f(r, \theta) = 0$ (although many curves will be represented by $r = f(\theta)$).

In fact, one use of polar coordinates is that some curves have relatively complicated equations in Cartesian coordinates but have much simpler equations in polar coordinates.

Ex. The circle of radius 3 in Cartesian coordinates is $x^2 + y^2 = 9$.

In polar coordinates, the equation of this circle is $r = 3$.

Ex. Sketch a graph of $r = 2\sin\theta$ by plotting points, then find the equivalent equation in Cartesian coordinates.



θ	$r = 2\sin\theta$
0	0
$\frac{\pi}{6}$	1
$\frac{\pi}{4}$	$\sqrt{2}$
$\frac{\pi}{3}$	$\sqrt{3}$
$\frac{\pi}{2}$	2
$\frac{2\pi}{3}$	$\sqrt{3}$
$\frac{3\pi}{4}$	$\sqrt{2}$
$\frac{5\pi}{6}$	1
π	0

$$y = r\sin\theta \quad \Rightarrow \quad \frac{y}{r} = \sin\theta$$

$$r = 2\sin\theta = \frac{2y}{r}$$

$$r^2 = 2y$$

$$x^2 + y^2 = 2y$$

$$x^2 + y^2 - 2y = 0$$

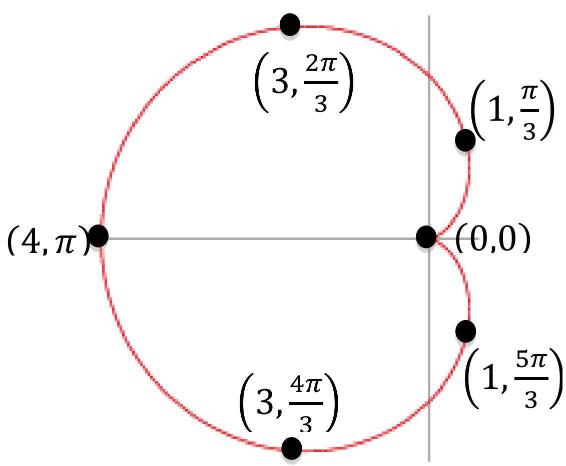
$$x^2 + y^2 - 2y + 1 = 1$$

$$x^2 + (y-1)^2 = 1.$$

This is a circle of radius 1 and center at $(0,1)$ in Cartesian coordinates.

Ex. Sketch a graph of $r = 2(1 - \cos\theta)$ by plotting points.

θ	$r = 2\,(1 - \cos\theta)$
0	0
$\frac{\pi}{3}$	1
$\frac{2\pi}{3}$	3
π	4
$\frac{4\pi}{3}$	3
$\frac{5\pi}{3}$	1



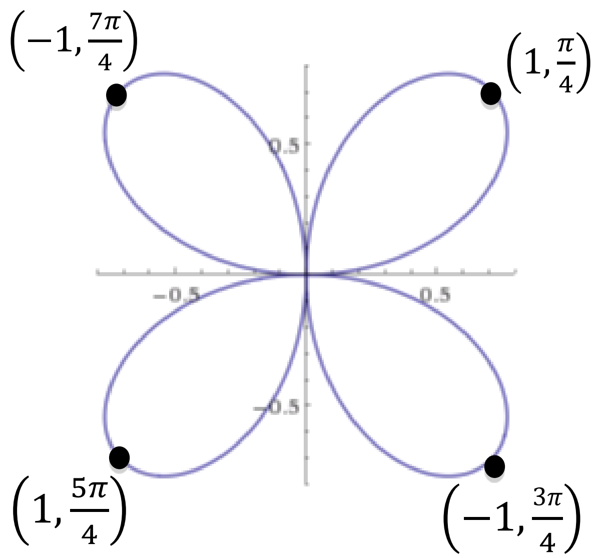
This curve is called a **cardioid**.

Note: In Cartesian coordinates this cardioid has the following equation

$$(x^2 + 2x + y^2)^2 = 4(x^2 + y^2).$$

Ex. Sketch a graph of $r = \sin 2\theta$ by plotting points.

θ	$r = \sin 2\theta$
0	0
$\frac{\pi}{4}$	1
$\frac{\pi}{2}$	0
$\frac{3\pi}{4}$	-1
π	0
$\frac{5\pi}{4}$	1
$\frac{3\pi}{2}$	0
$\frac{7\pi}{4}$	-1
2π	0



<u>Tangents to Polar Curves</u>

If $r = f(\theta)$, then:

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta\ .$$

Thus, we can say $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$ are parametric equations for the curve $r = f(\theta)$.

We know how to find $\frac{dy}{dx}$ (the slope of the tangent line to a curve) for parametric equations:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

We can find horizontal tangents when $\frac{dy}{d\theta} = 0$ (and $\frac{dx}{d\theta} \neq 0$).

We can find vertical tangents when $\frac{dx}{d\theta} = 0$ (and $\frac{dy}{d\theta} \neq 0$).

Ex. For the cardioid $r = 2(1 - \cos\theta)$:

a. Find the slope of the tangent line at $\theta = \frac{\pi}{4}$.

b. Find the points where the tangent line is horizontal or vertical.

a.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$\frac{dr}{d\theta} = 2\sin\theta$$

$$\frac{dy}{dx} = \frac{2\sin^2\theta + 2(1-\cos\theta)\cos\theta}{2\sin\theta\cos\theta - 2(1-\cos\theta)\sin\theta} = \frac{\sin^2\theta + (1-\cos\theta)\cos\theta}{2\sin\theta\cos\theta - \sin\theta}$$

At $\theta = \frac{\pi}{4}$ we have:

$$\frac{dy}{dx} = \frac{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(1-\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2}}{2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2}} = \frac{\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2-\sqrt{2}}.$$

b. To find horizontal tangents, we need to find where $\frac{dy}{d\theta} = 0$; $\left(\frac{dx}{d\theta} \neq 0\right)$.

To find vertical tangents, we need to find where $\frac{dx}{d\theta} = 0$; $\left(\frac{dy}{d\theta} \neq 0\right)$.

$$y = r\sin\theta = 2(1-\cos\theta)\sin\theta = 2(\sin\theta - \sin\theta\cos\theta)$$

$$\frac{dy}{d\theta} = 2\left(\cos\theta + \sin^2\theta - \cos^2\theta\right) = 0$$

$$\cos\theta + \sin^2\theta - \cos^2\theta = 0$$

$$\cos\theta + (1-\cos^2\theta) - \cos^2\theta = 0$$

$$-2\cos^2\theta + \cos\theta + 1 = 0$$

$$2\cos^2\theta - \cos\theta - 1 = 0$$

$$(2\cos\theta + 1)(\cos\theta - 1) = 0$$

$$\cos\theta = -\frac{1}{2} \text{ or } \cos\theta = 1$$

$$\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}.$$

$$x = r\cos\theta = 2(1-\cos\theta)\cos\theta = 2(\cos\theta - \cos^2\theta)$$

$$\frac{dx}{d\theta} = 2(-\sin\theta + 2\cos\theta\sin\theta) = 0$$

$$-\sin\theta + 2\cos\theta\sin\theta = 0$$

$$\sin\theta\,(-1 + 2\cos\theta) = 0$$

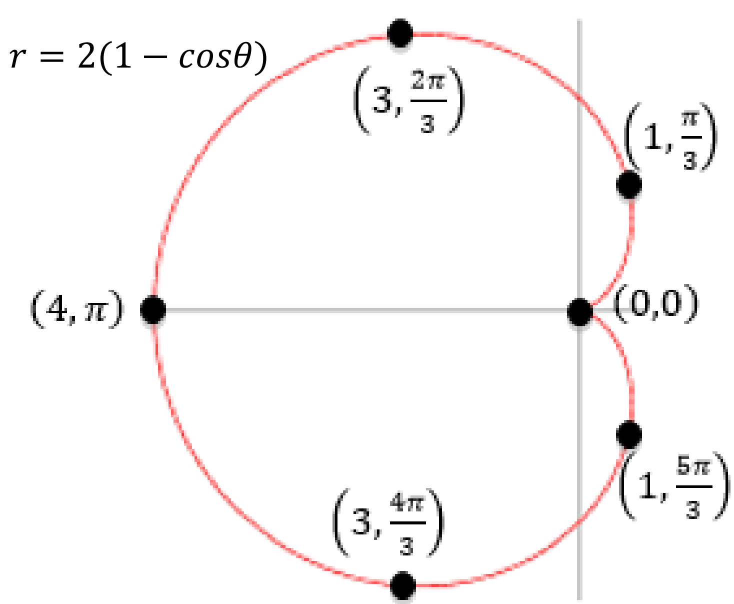
$$\sin\theta = 0 \quad \text{or} \quad \cos\theta = \frac{1}{2}$$

$$\theta = 0,\ \pi,\ \frac{\pi}{3},\ \frac{5\pi}{3}\ .$$

Notice that both $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ are 0 at $\theta = 0$. Otherwise, we can say that there are:

Horizontal tangents at: $\theta = \frac{2\pi}{3},\ r = 3$ and $\theta = \frac{4\pi}{3},\ r = 3.$

Vertical tangents at: $\theta = \frac{\pi}{3},\ r = 1,\ \ \theta = \frac{5\pi}{3},\ r = 1$ and $\theta = \pi,\ r = 4.$



To determine $\frac{dy}{dx}$ at $\theta = 0$:

$$\lim_{\theta\to 0}\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \lim_{\theta\to 0}\frac{2(\cos\theta + \sin^2\theta - \cos^2\theta)}{2(-\sin\theta + 2\cos\theta\sin\theta)}$$

Now divide the top and bottom by 2 and apply L'Hospital's Rule:

$$\frac{dy}{dx} = \lim_{\theta\to 0}\frac{-\sin\theta + 2sin\theta cos\theta + 2sin\theta cos\theta}{-\cos\theta + 2\cos^2\theta - 2\sin^2\theta} = \frac{0}{1} = 0\,.$$

So $\theta = 0,\ r = 0$ also has a horizontal tangent.