Parametric Curves in the Plane

One way to define a curve in the x- y plane is with an equation of the form:

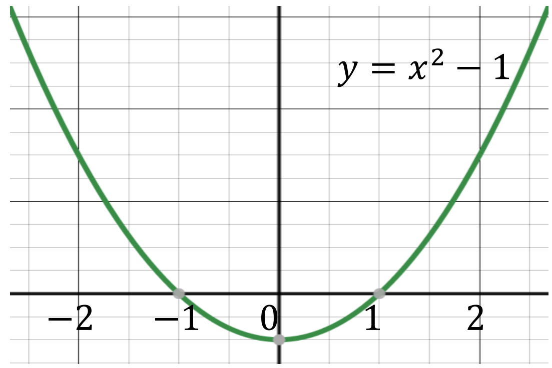
$$f(x, y) = 0$$

Ex.

$$y = x^2 - 1$$

OR

$$f(x, y) = y - x^2 + 1 = 0$$

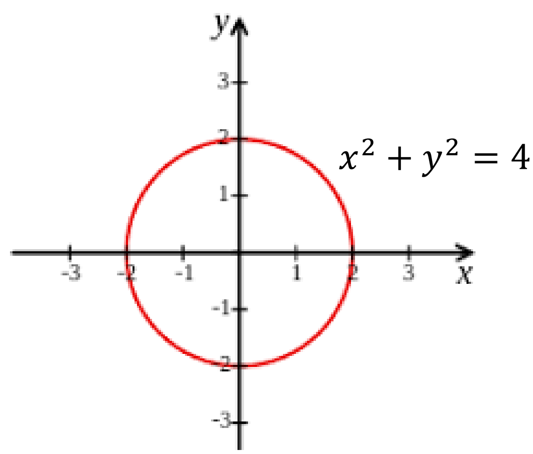


Ex.

$$x^2 + y^2 = 4$$

OR

$$f(x, y) = x^2 + y^2 - 4 = 0$$



One drawback of this approach can be seen when the curve is the path of an object. We might be able to describe the path of the object with an equation, $f(x, y) = 0$, but we don't know when the object is at a given point or how fast it's moving (or its acceleration).

A second way to define a curve in the x- y plane is with **parametric equations**.

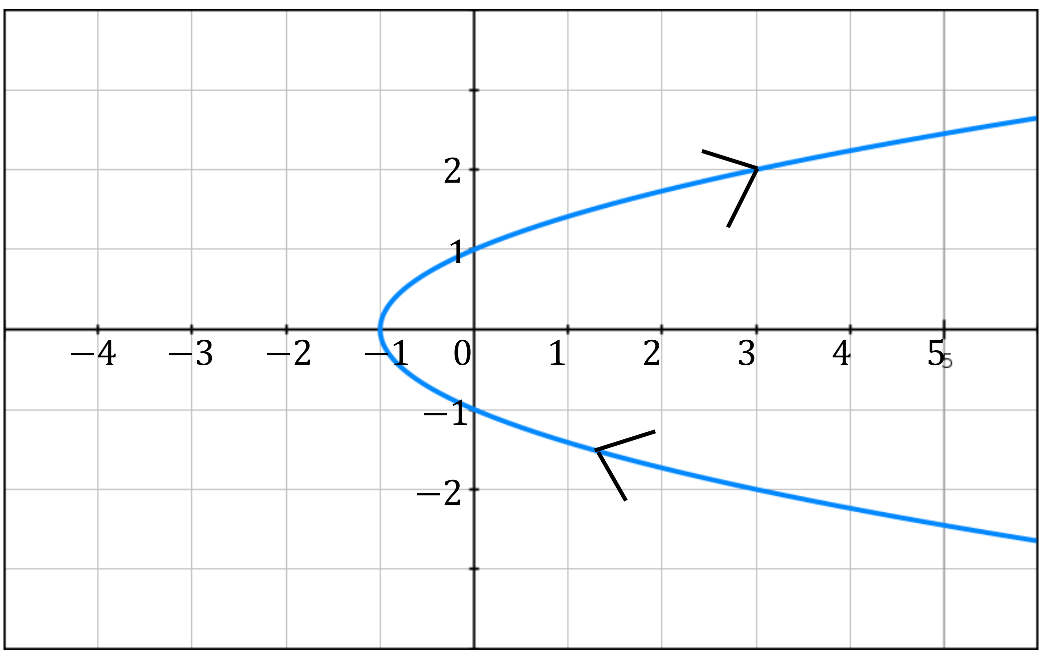
$$x = f(t); \quad y = g(t)$$

where t is on an interval I. In this case, t is called the **parameter**. The parameter t does not need to represent time, but if it does we can think of the curve defined by $x = f(t), y = g(t)$ as the path of an object where at any time $t \in I$ we know where the object is.

Ex. Sketch and identify the curve defined by the parametric equations below.

$$x = t^2 + 2t; \quad y = t + 1$$

t	x	y
-4	8	-3
-3	3	-2
-2	0	-1
-1	-1	0
0	0	1
1	3	2
2	8	3



The arrows show the direction the points are moving in as t increases.

In this case, we can identify the curve by 'eliminating' the parameter t and writing the curve as just a relationship between x and y.

$$
\begin{aligned}
x &= t^2 + 2t \\
y &= t + 1 \\
y - 1 &= t
\end{aligned}
$$

Now substitute for t in the first equation:

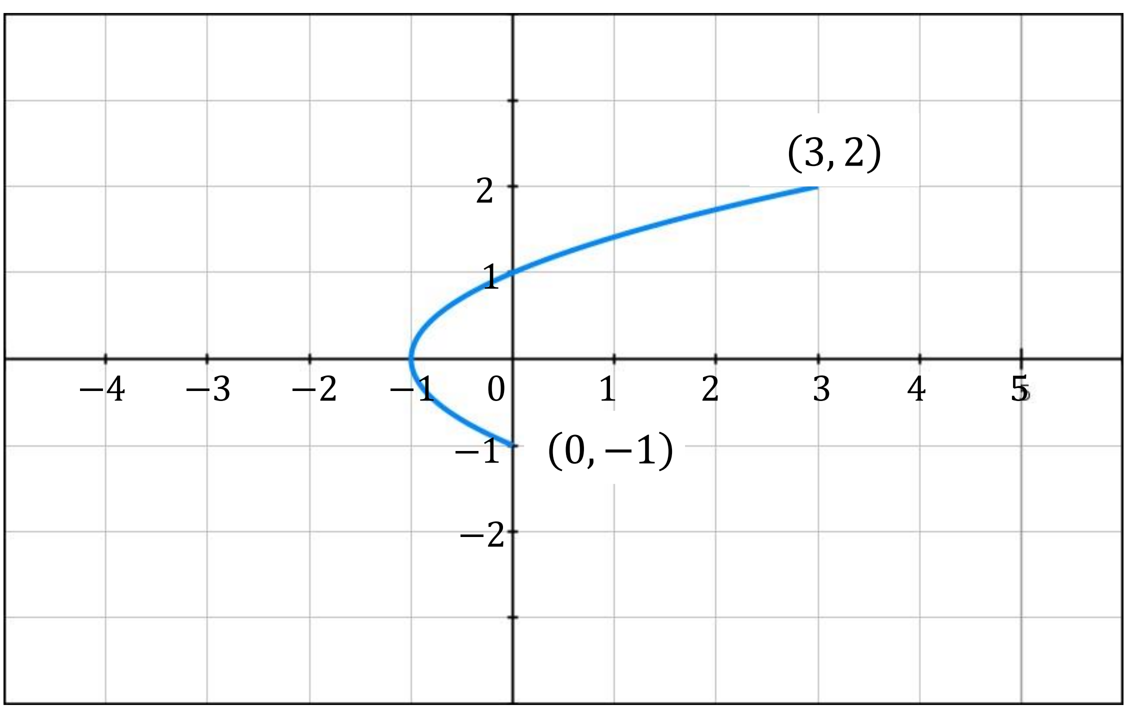
$$
\begin{aligned}
x &= (y-1)^2 + 2(y-1) \\
&= (y^2 - 2y + 1) + 2y - 2 \\
&= y^2 - 1
\end{aligned}
$$

Notice that $x = y^2 - 1$ is the equation of a parabola.

In this example, there was no restriction on the value of t. If we had said,

$$x = t^2 + 2t; \quad y = t + 1; \quad -2 \le t \le 1$$

then the curve would be:



In general, if we have: $x = f(t); \quad y = g(t); \quad a \le t \le b$, then we call $\big(f(a), g(a)\big)$ the initial point and $\big(f(b), g(b)\big)$ the terminal point.

Given any curve $y = f(x)$ there are an infinite number of ways to parametrize this curve. One way is:

$$x = t \qquad y = f(t)$$

Ex. Parametrize $y = x^2, \quad x \in \mathbb{R}$.

$$x = t \qquad y = t^2$$

Notice that:

$$x = t^3 \quad y = t^6$$

is also a parametrization of $y = x^2$.

However, the following is not a parametrization of the entire curve:

$$x = t^2 \qquad y = t^4$$

because $x \ge 0$ since $t^2 \ge 0$.

Similarly, given any curve $x = g(y)$ there are an infinite number of ways to parametrize this curve. One way is:

$$x = g(t) \qquad y = t$$

Ex. Parametrize $x = y^3 + y^2$

$$x = t^3 + t^2 \qquad y = t$$

Ex. Show that the circle $x^2 + y^2 = 1$ can be parametrized by

a. $x = \cos t\,, \quad y = \sin t\,, \quad 0 \le t \le 2\pi$

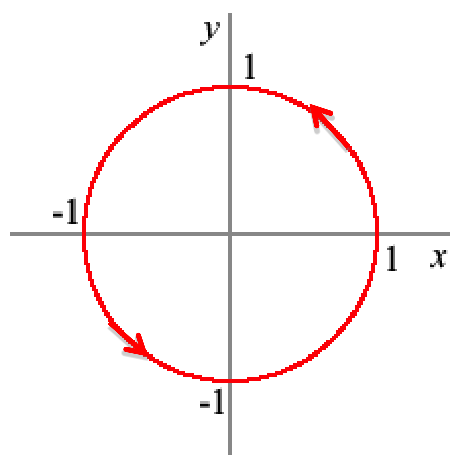
b. $x = \cos 2t\,, \quad y = \sin 2t\,, \quad 0 \le t \le \pi$

c. $x = \sin t\,, \quad y = \cos t\,, \quad 0 \le t \le 2\pi$

and find the differences between these paths.

a. $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

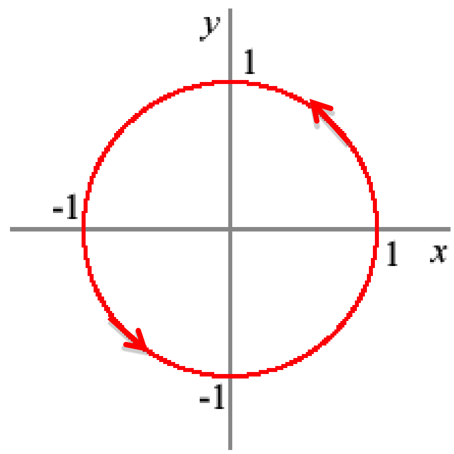
t	x	y
0	1	0
$\frac{\pi}{2}$	0	1
π	-1	0
$\frac{3\pi}{2}$	0	-1
2π	1	0



So $x = \cos t\,, \;\; y = \sin t$ is a parametrization of $x^2 + y^2 = 1$.

b. $x^2 + y^2 = \cos^2 2t + \sin^2 2t = 1$

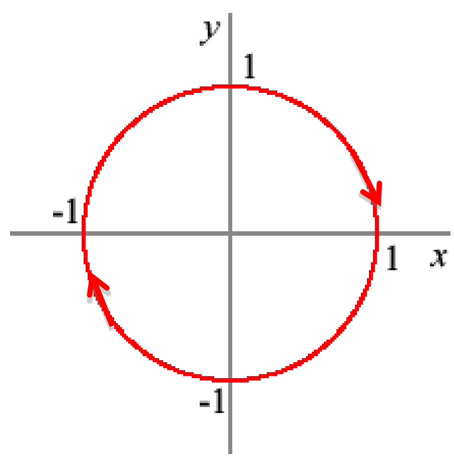
t	x	y
0	1	0
$\frac{\pi}{4}$	0	1
$\frac{\pi}{2}$	-1	0
$\frac{3\pi}{4}$	0	-1
π	1	0



Notice that this parametrization moves around the unit circle in the same direction as "a." but twice as fast.

c. $x^2 + y^2 = \sin^2 t + \cos^2 t = 1$

t	x	y
0	0	1
$\frac{\pi}{2}$	1	0
π	0	-1
$\frac{3\pi}{2}$	-1	0
2π	0	1



This parametrization goes around the unit circle in $t = 2\pi$, but it goes in the opposite direction from "a." and "b."

Ex. Find a parametrization of a circle of radius r and center (h, k).

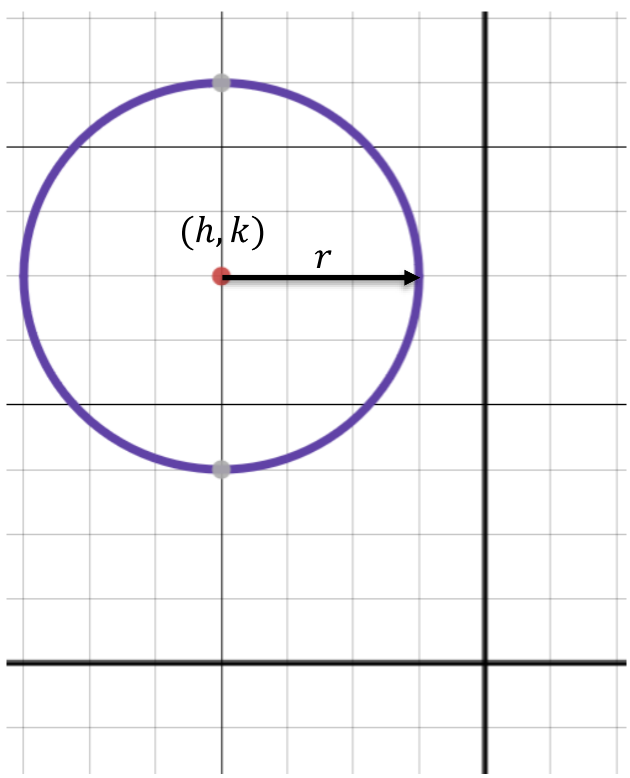
$$x = h + r\cos t \qquad 0 \le t \le 2\pi$$
$$y = k + r\sin t$$

Notice we can rewrite these as:

$$x - h = r\cos t$$
$$y - k = r\sin t$$

$$(x-h)^2 + (y-k)^2 = r^2\cos^2 t + r^2\sin^2 t = r^2$$
$$(x-h)^2 + (y-k)^2 = r^2$$

giving us the equation of a circle of radius r and center (h, k).

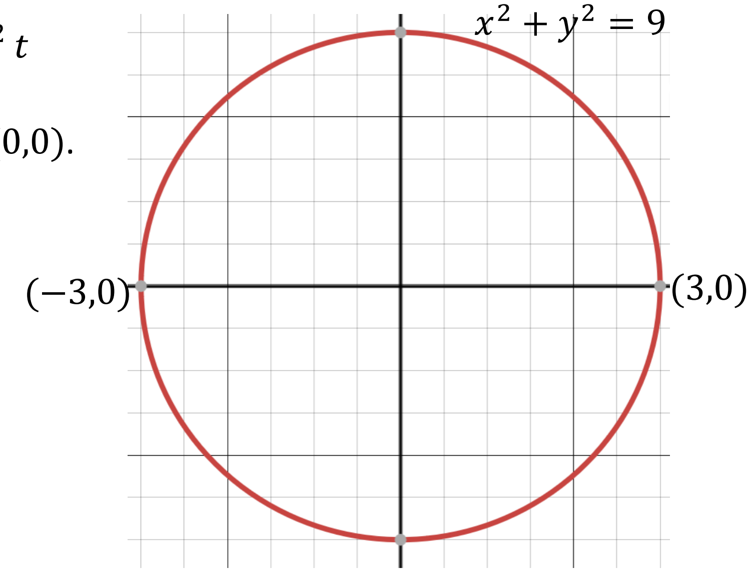


Ex. Identify the curves given by

a. $x = 3cost$
$y = 3sint$

b. $x = 4cost$
$y = 2sint$

a. $x^2 + y^2 = 9\cos^2 t + 9\sin^2 t$
$x^2 + y^2 = 9$
Circle of radius 3 centered at (0,0).



b. $\frac{x}{4} = cost$

$\frac{y}{2} = sint \quad \Longrightarrow \quad \left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = \cos^2 t + \sin^2 t = 1$

$\frac{x^2}{16} + \frac{y^2}{4} = 1;$ ellipse centered at (0,0), major axis is the x-axis.