The Alternating Series, Ratio, and Root Tests

The Alternating Series Test:

An **Alternating Series** is a series where the signs alternate in the sum.

Ex. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + (-1)^{n-1}\frac{1}{n} + \cdots$

This is called the "**Alternating Harmonic Series**".

Note: Just because the sum has a -1 raised to a power in front of the terms does NOT necessarily mean the series alternates. For example:

Ex. $\sum_{n=1}^{\infty}(-1)^{2n}(\frac{1}{n}) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots + \frac{1}{n} + \cdots$

Ex. $\sum_{n=1}^{\infty}(-1)^{2n+1}(\frac{1}{n}) = -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \cdots - \frac{1}{n} - \cdots$

Ex. $\sum_{n=1}^{\infty}(-1)^{n-1}(\sin(n)) = \sin(1) - \sin(2) + \sin(3) - \sin(4) + \cdots$

This is because $\sin(1) > 0, \sin(2) < 0, \sin(3) < 0, \sin(4) > 0$, etc.

You can also have an alternating series without a -1 to some power in front of the terms.

Ex. $\sum_{n=1}^{\infty}[\cos(n-1)\,\pi]\left(\frac{1}{n}\right)$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + (-1)^{n-1}\frac{1}{n} + \cdots$$

Alternating Series Test Theorem: If the alternating series

$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \cdots;$ $\qquad b_n > 0$, satisfies:

1. $b_{n+1} \leq b_n$ for all n (or at least from some n onward)
2. $\lim_{n\to\infty} b_n = 0$

then the series converges.

Although in the vast majority of the cases we will look at (and possibly all of them), when $\lim_{n\to\infty} b_n = 0$ it will also be true that $b_{n+1} \leq b_n$ for all n (or at least from some n onward). However, it is possible to have $\lim_{n\to\infty} b_n = 0$ and not have the sequence be decreasing. In that case, we couldn't apply the theorem.

So when you see an alternating series, you should think about the alternating series test.

Ex. Determine the convergence of the alternating harmonic series:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + (-1)^{n-1}\frac{1}{n} + \cdots$$

This series is alternating.

$b_n = \frac{1}{n};\ \lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n} = 0$ and

$b_{n+1} = \frac{1}{n+1} \leq \frac{1}{n} = b_n$ for all n.

Hence, the alternating series test says this series converges.

Ex. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^3+1}$.

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1} = \frac{1}{2} - \frac{4}{9} + \frac{9}{28} + \cdots$ is an alternating series.

$$\lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{n^2}{n^2}\left(\frac{1}{n+\frac{1}{n^2}}\right) = 0.$$

$$f(x) = \frac{x^2}{x^3+1}$$

$$f'(x) = \frac{(x^3+1)(2x) - x^2(3x^2)}{(x^3+1)^2} = \frac{x(2-x^3)}{(x^3+1)^2}$$

$2 - x^3 < 0$ if $x > \sqrt[3]{2}$, so for $x > \sqrt[3]{2}$ we know $\frac{x(2-x^3)}{(x^3+1)^2} < 0.$

Thus, $f(x)$ is decreasing for $x > \sqrt[3]{2}$ and the sequence $\frac{n^2}{n^3+1}$ is decreasing for $n > \sqrt[3]{2}$.

So the alternating series theorem applies and $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^3+1}$ converges.

We can actually determine how close the partial sums, S_n, of an alternating series are to the total sum without knowing what the total sum is.

Alternating Series Estimation Theorem: If $S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies:

1. $b_{n+1} \le b_n$
2. $\lim_{n\to\infty} b_n = 0$

then $|S - S_n| = |R_n| \le b_{n+1}$.

Ex. Show that $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$ is convergent and determine how many terms of the series are needed to find the sum to within 0.01 .

First, let's show that $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$ is convergent with the alternating series test.

$\sum_{n=1}^{\infty} \frac{(-2)^n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n!}$ is an alternating series.

Claim: $\lim_{n\to\infty} \frac{2^n}{n!} = 0$.

$$0 \le \frac{2^n}{n!} = \frac{2}{n}\left(\frac{2}{n-1} \cdot \ldots \cdot \frac{2}{2} \cdot \frac{2}{1}\right) \le \frac{2}{n}(2) = \frac{4}{n}$$

as every factor is $\frac{2}{n-1}, \ldots, \frac{2}{2} \le 1$.

Now, by the squeeze theorem since $\lim_{n\to\infty} \frac{4}{n} = 0 \Rightarrow \lim_{n\to\infty} \frac{2^n}{n!} = 0$.

By the same kind of argument, $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for any constant a.

Claim: $b_{n+1} \le b_n$.

$$b_{n+1} = \frac{2^{n+1}}{(n+1)!} = \frac{(2)(2^n)}{(n+1)(n!)} = \frac{2}{n+1}(b_n) \le b_n$$

since for $n \ge 1$, $\frac{2}{n+1} \le 1$.

Thus, we have $b_{n+1} \leq b_n$ for all $n \geq 1$.

So by the alternating series test $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$ converges.

The error in the series after n terms is expressed as:

$$|S - S_n| = |R_n| \leq b_{n+1} = \frac{2^{n+1}}{(n+1)!} \leq 0.01$$

By trial and error we can find an n such that $\frac{2^{n+1}}{(n+1)!} \leq 0.01$.

n	$\frac{2^{n+1}}{(n+1)!}$
5	$\frac{2^6}{6!} \approx 0.089$
6	$\frac{2^7}{7!} \approx 0.025$
7	$\frac{2^8}{8!} \approx 0.006$

We can see that $0.006 \leq 0.01$, so:

$$S \approx S_7 = -2 + \frac{2^2}{2!} - \frac{2^3}{3!} + \frac{2^4}{4!} - \frac{2^5}{5!} + \frac{2^6}{6!} - \frac{2^7}{7!}$$

within an error of 0.01 .

Note: If the problem said find the number of terms that are needed to find the sum so that it's accurate to the third decimal place, it would mean $|R_n| \leq 0.0005.$ Also, this method only works in general for alternating series.

Absolute Convergence

For any series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots$, we can create a new series.

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + |a_4| + \cdots$$

Def. A series $\sum_{n=1}^{\infty} a_n$ is called **Absolutely Convergent** if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ converges.

Note: If $\{a_n\}$ are already ≥ 0, absolute convergence means the same thing as convergence.

Ex. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^5} = 1 - \frac{1}{2^5} + \frac{1}{3^5} - \frac{1}{4^5} + \cdots$ is absolutely convergent because $\sum_{n=1}^{\infty} \frac{1}{n^5} = 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \cdots$ is convergent since it's a p-series with $p = 5 > 1$.

Ex. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + (-1)^{n-1} \frac{1}{n} + \cdots$, called the alternating harmonic series, is convergent (by the alternating series test), but NOT absolutely convergent because:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots + \frac{1}{n} + \cdots,$$

the harmonic series, is divergent.

Def. If a series is convergent but not absolutely convergent, then it is called **Conditionally Convergent**.

Ex. The alternating harmonic series, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$, is conditionally convergent.

Theorem: If a series is absolutely convergent then it is convergent.

Ex. Determine the convergence of $\sum_{n=1}^{\infty}\frac{\sin(2n+1)}{n^2}$.

Notice that $|\sin(2n+1)| \leq 1$ so $\left|\frac{\sin(2n+1)}{n^2}\right| \leq \frac{1}{n^2}$.

We know $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges because it's a p-series with $p = 2 > 1$.

Thus, $\sum_{n=1}^{\infty}\left|\frac{\sin(2n+1)}{n^2}\right|$ converges by the comparison test.

So $\sum_{n=1}^{\infty}\frac{\sin(2n+1)}{n^2}$ is absolutely convergent (and hence also convergent).

Ex. Determine if the following series is convergent or divergent. If convergent, is it absolutely convergent?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)} = -\frac{1}{ln2} + \frac{1}{ln3} - \frac{1}{ln4} + \frac{1}{ln5} - \cdots$$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$ is convergent by the alternating series test since:

a) It's an alternating series.
b) $\lim_{n\to\infty} \frac{1}{\ln(n)} = 0$ since $\lim_{n\to\infty} \ln n = \infty$.
c) $b_{n+1} = \frac{1}{\ln(n+2)} \leq \frac{1}{\ln(n+1)} = b_n$ for all $n \geq 1$.

This series is not absolutely convergent because $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ diverges by the comparison test (hence it is conditionally convergent).

$\ln(n+1) \leq n+1 \;\Rightarrow$ for $n \geq 1 \;\Rightarrow\; \frac{1}{\ln(n+1)} \geq \frac{1}{n+1}$ but $\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges by the integral test since:

a) $f(x) = \frac{1}{x+1} \;\Rightarrow\; f'(x) = -\frac{1}{(x+1)^2} < 0$ for $x \geq 1$, so $\left\{\frac{1}{n+1}\right\}$ is decreasing.

b) $\int_1^{\infty} \frac{1}{x+1}\,dx = \lim_{n\to\infty} \int_1^{b} \frac{1}{x+1}\,dx$
$= \lim_{b\to\infty} (\ln(b+1) - \ln(2)) = \infty$.

So $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ diverges by the comparison test.

The Ratio Test

The Ratio Test can be very useful for determining absolute convergence.

The Ratio Test Theorem:

1. If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

2. If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

3. If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L = 1$, then the ratio test is inconclusive (the series might converge or it might diverge. We need to use some other method).

Notice:

If we apply the ratio test to $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which we already know converges (because it's a p-series with $p = 2 > 1$), we get:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1,$$ so the ratio test is inconclusive.

If we apply the ratio test to $\sum_{n=1}^{\infty} \frac{1}{n}$, which we know diverges (why?) we get:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1,$$ so the ratio test is again inconclusive.

The ratio test is often useful when a_n has an $n!$ term or $(f(n))^n$ term.

Ex. Test the convergence of $\sum_{n=1}^{\infty} \frac{2^n}{n!}$.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n\to\infty} \left(\frac{2^{n+1}}{(n+1)!}\right)\left(\frac{n!}{2^n}\right)$$

$$= \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1 .$$

So we can say $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ is absolutely convergent by the ratio test.

Ex. Test the convergence of $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \lim_{n\to\infty} \left(\frac{(n+1)^{n+1}}{(n+1)!}\right)\left(\frac{n!}{n^n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)(n+1)^n}{(n+1)(n)!}\right)\left(\frac{n!}{n^n}\right) = \lim_{n\to\infty} \frac{(n+1)^n}{n^n}$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e > 1 .$$

So the series diverges by the ratio test since $a_n \geq 0$ for all n.

Ex. Determine the convergence of $\sum_{n=1}^{\infty} \frac{n^2(2^{n+1})}{3^n}$.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{(n+1)^2 2^{n+2}}{3^{n+1}}}{\frac{n^2 2^{n+1}}{3^n}} = \lim_{n\to\infty} \left(\frac{(n+1)^2 2^{n+2}}{3^{n+1}}\right)\left(\frac{3^n}{n^2 2^{n+1}}\right)$$

$$= \lim_{n\to\infty} \left(\frac{2}{3}\right)\left(\frac{(n+1)^2}{n^2}\right) = \lim_{n\to\infty} \frac{2}{3}\left(\frac{n^2+2n+1}{n^2}\right)$$

$$= \lim_{n\to\infty} \left(\frac{2}{3}\right)\left(\frac{n^2}{n^2}\right)\left(\frac{1+\frac{2}{n}+\frac{1}{n^2}}{1}\right) = \frac{2}{3} < 1.$$

So the series is absolutely convergent by the ratio test.

Rate of Growth from slowest to fastest:

1. $\ln(n)$ 2. $n^k;\ k > 1$ 3. $a^n\ ;\ a > 1$ 4. $n!$ 5. n^n

$\sum_{n=1}^{\infty} \frac{lower\ \#}{higher\ \#}$ converges, $\sum_{n=1}^{\infty} \frac{higher\ \#}{lower\ \#}$ diverges.

Ex. $\sum_{n=1}^{\infty} \frac{n^{100}}{(1.01)^n}$ converges and $\sum_{n=1}^{\infty} \frac{(1.01)^n}{n^{100}}$ diverges

Convergence/divergence of both of these examples can be shown with the Ratio Test.

<u>The Root Test</u>

The Root Test can be useful when n is in the exponent of a_n, other than $(-1)^n$.

The Root Test Theorem:

1. If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

2. If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

3. If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$, then the root test is inconclusive.

Ex. Test the convergence of $\sum_{n=1}^{\infty} (\frac{2n}{5n+3})^n$.

$$\lim_{n\to\infty} \sqrt[n]{(\frac{2n}{5n+3})^n} = \lim_{n\to\infty} \left(\left(\frac{2n}{5n+3}\right)^n\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{2n}{5n+3}\right) = \frac{2}{5} < 1$$

by L'Hospital's Rule, so the series converges absolutely by the root test.

Ex. Test the convergence of $\sum_{n=1}^{\infty}\left(\frac{-3n}{2n+1}\right)^{3n}$.

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{(-3n)}{2n+1}\right|^{3n}} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{3n}{2n+1}\right)^{3n}} = \lim_{n\to\infty}\left(\left(\frac{3n}{2n+1}\right)^{3n}\right)^{\frac{1}{n}}$$

$$= \left(\frac{3}{2}\right)^{3} = \frac{27}{8} > 1.$$

So the series is divergent by the root test

Note: This series is also divergent by the divergence test.

Ex. Test the convergence of $\frac{1}{(\ln 3)^3} + \frac{1}{(\ln 4)^4} + \frac{1}{(\ln 5)^5} + \frac{1}{(\ln 6)^6} + \cdots$

$$\lim_{n\to\infty}\sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n\to\infty}\left(\frac{1}{(\ln n)^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{\ln n} = 0\ .$$

So the series is absolutely convergent by the root test.

Summary of Convergence/Divergence Tests for Series

Test	Series	Conditions on Convergence	Conditions on Divergence	Comments
Divergence Test	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	Test can only prove divergence
Geometric Series	$\sum_{n=1}^{\infty} ar^{n-1}$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	$S = \frac{a}{1-r}$
p-series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	$p = 1$ is Harmonic Series
Alternating Series Test	$\sum_{n=1}^{\infty} (-1)^n a_n$	0$< a_{n+1} \leq a_n$ $\lim_{n\to\infty} a_n = 0$		R_n, error after n-terms $\lvert R_n\rvert \leq a_{n+1}$
Integral Test	$\sum_{n=1}^{\infty} a_n$ $f(n) = a_n, f > 0$ is Decreasing/Cont.	$\int_1^{\infty} f(x)dx\ conv.$	$\int_1^{\infty} f(x)dx\ div.$	Must be able to calculate $\int_1^{\infty} f(x)dx$
Ratio Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \lvert\frac{a_{n+1}}{a_n}\rvert < 1$	$\lim_{n\to\infty} \lvert\frac{a_{n+1}}{a_n}\rvert > 1$	Limit=1 is Inconclusive. Useful: a_n has $n!$ term, n in an exponent
Root Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$	Limit=1 is Inconclusive. Useful: a_n has an n in exponent

Test	Series	Conditions on Convergence	Conditions on Divergence	Comments
Comparison Test	$\sum_{n=1}^{\infty} a_n$	$0 \leq a_n \leq b_n$ $\sum_{n=1}^{\infty} b_n$ converges	$0 \leq b_n \leq a_n$ $\sum_{n=1}^{\infty} b_n$ $diverges$	Useful when $\sum_{n=1}^{\infty} a_n$ is similar p-series or Geometric series
Limit Comp. Test	$\sum_{n=1}^{\infty} a_n$ $a_n \geq 0$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ $\sum_{n=1}^{\infty} b_n$ converges $0 \leq b_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ $\sum_{n=1}^{\infty} b_n$ diverges $0 \leq b_n$	Useful when $\sum_{n=1}^{\infty} a_n$ similar to p-series or Geometric series

Ex. In each case state which method you would use to determine if the series converges absolutely, conditionally, or diverges.

a. $\sum_{k=1}^{\infty} \frac{(-1)^k (5k)}{3k^2-1}$.

Alternating series test for convergence and comparison test with $\sum_{k=1}^{\infty} \frac{5}{3k}$ to show it's not absolutely convergent.

b. $\sum_{j=1}^{\infty} \frac{(-1)^j(\sin(j))}{j^2}$.

Comparison test with $\sum_{j=1}^{\infty} \frac{1}{j^2}$ to show absolute convergence (and thus convergence)

c. $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n!}\right)$.

Divergence test since $\lim_{n\to\infty} \cos\left(\frac{1}{n!}\right) = 1 \neq 0$.

d. $\sum_{j=2}^{\infty} \frac{(-1)^j}{j\sqrt{\ln(j)}}$.

Alternating series test for convergence and integral test to show it's not absolutely convergent.

e. $\sum_{n=1}^{\infty} \frac{5(-3)^n}{2^{2n}}$.

Ratio test to show the series is absolutely convergent.

f. $\sum_{n=1}^{\infty} \frac{(-1)^n(2^n+1)}{2^{n+1}}$.

The series diverges by the divergence test since $\lim_{n\to\infty} \frac{(-1)^n(2^n+1)}{2^{n+1}}$ does not exist and hence is not equal to 0.