# Geometric and Telescoping Series

If we add the terms of a sequence we get a **series**:

Sequence: $a_{1,}\ a_{2,}\ a_{3,}\ a_{4,}\ a_{5,}\ \ldots$

Series: $a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$

Key Question: When does the sum of an infinite number of terms have a finite answer?

Answer: Given a series $\sum_{i=1}^{\infty} a_i$, we add up the first $n$ terms and call this $S_n$:

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots + a_n.$$

If the sequence $\{S_n\}$ converges to $S < \infty$ i.e., $\lim_{n\to\infty} S_n = S$, then we call $S$ the sum of the series and we say the series **converges** to $S$.

If the sequence $\{S_n\}$ doesn't converge, we say the series **diverges**.

Ex. $\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots$

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

$\vdots$

$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$

$$\lim_{n\to\infty} \frac{2^n - 1}{2^n} = \lim_{n\to\infty} \frac{2^n(1 - \frac{1}{2^n})}{2^n} = 1.$$

Thus we say that $\sum_{i=1}^{\infty} \frac{1}{2^i}$ converges, and it converges to $1$.

In most series we will deal with, even if it converges, we won't be able to tell what number it converges to. However, there is a class of series, called geometric series, where we will be able to determine what number a convergent series converges to.

Def. A **geometric series** has the form:

$$a + ar + ar^2 + ar^3 + ar^4 + ar^5 + \cdots ar^n + \cdots = \sum_{i=1}^{\infty} ar^{i-1}.$$

Notice for a geometric series, to get from one term, say $ar^3$, to the next term, $ar^4$, you always multiply by the same number, $r$.

Ex. $\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots$ is a geometric series with $a = \frac{1}{2}$, and $r = \frac{1}{2}$.

Ex. $\sum_{i=1}^{\infty} (-1)^i (\frac{2}{3})^i = -\frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81} + \cdots + (-1)^n (\frac{2}{3})^n + \cdots$ is a geometric series with $a = -\frac{2}{3}$, and $r = -\frac{2}{3}$.

Ex. $\sum_{i=1}^{\infty} (\frac{2}{3})^{i+2} = \left(\frac{2}{3}\right)^3 + (\frac{2}{3})^4 + (\frac{2}{3})^5 + (\frac{2}{3})^6 + \cdots + (\frac{2}{3})^{n+2} + \cdots$

$= \frac{8}{27} + \frac{16}{81} + \frac{32}{243} + \frac{64}{729} + \cdots$ is a geometric series with $a = \frac{8}{27}$ and $r = \frac{2}{3}$.

Ex. $\sum_{i=1}^{\infty} 2(10)^{i-1} = 2 + 20 + 200 + 2000 + \cdots + 2(10)^{n-1} + \cdots$ is a geometric series with $a = 2$ and $r = 10$.

Sum of an Infinite Geometric Series

For a geometric series, $\sum_{i=1}^{\infty} ar^{i-1}$, we have:

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + ar^5 + \cdots ar^{n-1}$$

$$rS_n = \quad ar + ar^2 + ar^3 + ar^4 + ar^5 + \cdots ar^{n-1} + ar^n$$

$$S_n - rS_n = a - ar^n = a(1 - r^n)$$

$$S_n(1 - r) = a(1 - r^n)$$

$$S_n = \frac{a(1-r^n)}{(1-r)} \ .$$

The geometric series, $\sum_{i=1}^{\infty} ar^{i-1}$, converges when the sequence $\{S_n\}$ converges. That is when:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{(1-r)} \text{ converges.}$$

If $-1 < r < 1$, then we have:

$$\lim_{n\to\infty} \frac{a(1-r^n)}{(1-r)} = \frac{a}{1-r} \, .$$

Thus for an infinite geometric series we have:

$\sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1-r}$ if $|r| < 1$, i.e. the series converges.

If $|r| \geq 1$ i.e., if $r \geq 1$ or $r \leq -1$, the infinite geometric series diverges.

Ex. $\sum_{i=1}^{\infty} (-1)^i (\frac{2}{3})^i = -\frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81} + \cdots + (-1)^n (\frac{2}{3})^n + \cdots$ is a geometric series with $a = -\frac{2}{3}$, and $r = -\frac{2}{3}$. Since $|r| < 1$, this series converges to:

$$S = \frac{a}{1-r} = \frac{-\frac{2}{3}}{1-(-\frac{2}{3})} = \frac{-\frac{2}{3}}{\frac{5}{3}} = \left(-\frac{2}{3}\right)\left(\frac{3}{5}\right) = -\frac{2}{5} \, .$$

Ex. Determine if the series converges or diverges. If it converges, what does it converge to?

a. $\sum_{n=1}^{\infty}(\sin 4)^n$

b. $\sum_{n=1}^{\infty}(3)^{2n-1}\,(4)^{1-2n}$

a. $\sum_{n=1}^{\infty}(\sin 4)^n = (\sin 4) + (\sin 4)^2 + (\sin 4)^3 + \cdots + (\sin 4)^n + \cdots$

This is a geometric series with $a = \sin 4$ and $r = \sin 4$.

Since $|\sin 4| < 1$, the series converges and:

$$\sum_{n=1}^{\infty}(\sin 4)^n = \frac{a}{1-r} = \frac{\sin 4}{1-\sin 4}.$$

b. $\sum_{n=1}^{\infty}(3)^{2n-1}\,(4)^{1-2n} = \sum_{n=1}^{\infty}\frac{1}{3}(3)^{2n}\cdot 4(4)^{-2n}$

$$= \sum_{n=1}^{\infty}\frac{4}{3}\left(\frac{3}{4}\right)^{2n}$$

$$= \frac{4}{3}\left(\frac{3}{4}\right)^2 + \frac{4}{3}\left(\frac{3}{4}\right)^4 + \frac{4}{3}\left(\frac{3}{4}\right)^6 + \frac{4}{3}\left(\frac{3}{4}\right)^8 + \cdots$$

Thus, this is a geometric series with

$$a = \frac{4}{3}\left(\frac{3}{4}\right)^2 = \frac{3}{4}; \quad r = \left(\frac{3}{4}\right)^2 = \frac{9}{16}.$$

Since $|r| = \frac{9}{16} < 1$ the series converges and:

$$\sum_{n=1}^{\infty}(3)^{2n-1}(4)^{1-2n} = \frac{a}{1-r} = \frac{\frac{3}{4}}{1-\frac{9}{16}} = \frac{\frac{3}{4}}{\frac{7}{16}} = \frac{3}{4}\cdot\frac{16}{7} = \frac{12}{7}.$$

Ex. Write $3.621212121\ldots = 3.6\overline{21}$ as a ratio of integers.

$$3.621212121\ldots = 3.6 + \frac{21}{10^3} + \frac{21}{10^5} + \frac{21}{10^7} + \cdots$$

After the $3.6$ we have a geometric series with $a = \frac{21}{10^3}$ and $r = \frac{1}{10^2}$.

Since $|r| = \frac{1}{10^2} < 1$ the series converges.

Thus we can write:

$$3.6 + \frac{21}{10^3} + \frac{21}{10^5} + \frac{21}{10^7} + \cdots = 3.6 + \frac{\frac{21}{10^3}}{1 - \frac{1}{10^2}}$$

$$= 3.6 + \frac{\left(\frac{21}{1000}\right)}{\frac{99}{100}}$$

$$= \frac{36}{10} + \frac{21}{990}$$

$$= \frac{239}{66}.$$

There is another class of series (NOT geometric series) where we can find the infinite sum. These are called Telescoping Series.

Ex. Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \frac{1}{4\cdot5} + \cdots + \frac{1}{n(n+1)} + \cdots$

The trick is to use partial fractions.

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{A(n+1)+B(n)}{n(n+1)}.$$

The numerators have to be equal so:

$1 = A(n+1) + B(n)$ ; This is true for all numbers $n$, so in particular:

$n = 0$ means $1 = A$

$n = -1$ means $1 = -B \; or \; B = -1$ .

Thus we have:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

This means that:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) + \cdots$$

So $S_n = 1 - \frac{1}{n+1}$.

Thus, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$

So this means that:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \; .$$

Ex. $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the **Harmonic Series**. This is a very important series. Show that the harmonic series is divergent. Notice that since the harmonic series is divergent, then any non-zero multiple of the harmonic series is also divergent (e.g. $\sum_{n=1}^{\infty} \frac{1}{100n}$ is divergent).

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) > 1 + \frac{2}{2}$$

$$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$
$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2}$$

$$S_{16} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right)$$
$$> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{4}{2}.$$

$S_{2^n} > 1 + \frac{n}{2}$, which shows $S_{2^n} \to \infty$ as $n \to \infty$. Thus, the harmonic series diverges.

Theorem: If the series $\sum_{i=1}^{\infty} a_i$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Proof: Let $S_n = a_1 + \cdots + a_n$.

Then $a_n = S_n - S_{n-1}$, since we know $\sum_{i=1}^{\infty} a_i$ converges.

$\lim_{n\to\infty} S_n = S$ and $\lim_{n\to\infty} S_{n-1} = S$.

Thus, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = 0$.

Divergence Test Theorem:

If you have a series $\sum_{i=1}^{\infty} a_i$ and $\lim_{n\to\infty} a_n \neq 0$, then

$\sum_{i=1}^{\infty} a_i$ diverges.

NOTE: You can **<u>never</u>** use the divergence test to show a series converges, only to show a series diverges.

The fact that $\lim_{n\to\infty} a_n = 0$ for a series $\sum_{i=1}^{\infty} a_i$ tells us <u>nothing</u> about whether the series converges of diverges:

$\sum_{i=1}^{\infty} \frac{1}{i}$ has $\lim_{n\to\infty} a_n = 0$ and it diverges.

$\sum_{j=1}^{\infty} \frac{1}{2^j}$ also has $\lim_{n\to\infty} a_n = 0$, but it converges.

Ex. Show $\sum_{n=1}^{\infty} \frac{2n^3}{3n^3+1}$ diverges.

$$\lim_{n\to\infty} \frac{2n^3}{3n^3+1} = \lim_{n\to\infty} \frac{n^3(2)}{n^3\left(3+\frac{1}{n^3}\right)} = \frac{2}{3} \neq 0.$$

$\lim_{n\to\infty} a_n = \frac{2}{3} \neq 0 \Rightarrow \sum_{n=1}^{\infty} \frac{2n^3}{3n^3+1}$ diverges by the divergence theorem.

Theorem: If $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ are convergent series, then so are $\sum_{i=1}^{\infty} ca_i$ ($c$ a constant), $\sum_{i=1}^{\infty}(a_i + b_i)$ , $\sum_{i=1}^{\infty}(a_i - b_i)$ , and:

1. $\sum_{i=1}^{\infty} ca_i = c\sum_{i=1}^{\infty} a_i$
2. $\sum_{i=1}^{\infty}(a_i + b_i) = \sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i$
3. $\sum_{i=1}^{\infty}(a_i - b_i) = \sum_{i=1}^{\infty} a_i - \sum_{i=1}^{\infty} b_i$

The previous theorem follows from the limit laws for sequences applied to the partial sums, $S_n$, of the series.

Ex. Find $\sum_{n=4}^{\infty} \frac{2}{(n^2-4n+3)}$ .

$$\sum_{n=4}^{\infty} \frac{2}{(n^2-4n+3)} = \sum_{n=4}^{\infty} \frac{2}{(n-3)(n-1)}$$

$$\frac{2}{(n-3)(n-1)} = \frac{A}{n-3} + \frac{B}{n-1} = \frac{A(n-1)+B(n-3)}{(n-3)(n-1)}$$

$$2 = (A + B)n + (-A - 3B)$$

$$A + B = 0$$

$$-A - 3B = 2$$

$$B = -A$$

$$-A + 3A = 2$$

$$2A = 2$$

$$A = 1$$

$$B = -1$$

$$\sum_{n=4}^{\infty} \frac{2}{(n^2-4n+3)} = \sum_{n=4}^{\infty} \left(\frac{1}{n-3} - \frac{1}{n-1}\right)$$

$$= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{n-3} - \frac{1}{n-1}\right) + \cdots$$

$$S_n = \frac{1}{1} + \frac{1}{2} - \frac{1}{n-2} - \frac{1}{n-1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1}{1} + \frac{1}{2} - \frac{1}{n-2} - \frac{1}{n-1}\right) = \frac{3}{2} .$$

So: $\sum_{n=4}^{\infty} \frac{2}{(n^2-4n+3)} = \frac{3}{2}$ .

Ex. Find $\sum_{n=2}^{\infty} \frac{2^{n-1}-4(3)^n}{4^n}$

$$\sum_{n=2}^{\infty} \frac{2^{n-1}-4(3)^n}{4^n} = \sum_{n=2}^{\infty} \frac{2^{n-1}}{4^n} - \sum_{n=2}^{\infty} \frac{4(3)^n}{4^n}$$

$$= \sum_{n=2}^{\infty} \frac{2^{-1}(2)^n}{4^n} - \sum_{n=2}^{\infty} 4\left(\frac{3}{4}\right)^n$$

$$= \frac{1}{2}\sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n - 4\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n$$

$$\sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots; \quad \sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n = \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \cdots$$

$$a = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \qquad\qquad a = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$$

$$r = \frac{1}{2} \qquad\qquad r = \frac{3}{4}$$

$$\sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n = \frac{a}{1-r} = \frac{\frac{1}{4}}{1-\frac{1}{2}} = \frac{1}{2} \qquad\qquad \sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n = \frac{a}{1-r} = \frac{\frac{9}{16}}{1-\frac{3}{4}} = \frac{9}{4}$$

$$\sum_{n=2}^{\infty} \frac{2^{n-1}-4(3)^n}{4^n} = \frac{1}{2}\left(\frac{1}{2}\right) - 4\left(\frac{9}{4}\right) = \frac{1}{4} - 9 = -\frac{35}{4}\,.$$