Sequences

A sequence is a list of numbers:

$$a_{1,}\ a_{2,}\ a_{3,}\ a_{4,}\ \dots,\ a_n, \dots$$

This sequence may be denoted by $\{a_{1,}\ a_{2,}\ a_{3,}\ a_{4,}\ \dots,\ a_n, \dots\}$, $\{a_n\}$, or just a_n.

Ex.

a. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots$ where $a_n = 1/n$; or $\{1/n\}$

b. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \dots$ in this sequence $a_n = (-1)^{n-1}(\frac{1}{2^{n-1}})$; or

$$\{a_n\} = \{(-1)^{n-1}(\frac{1}{2^{n-1}})\}$$

c. $\sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \dots$ in this sequence $a_n = \sqrt{n+4}$; or

$$\{a_n\} = \{\sqrt{n+4}\}$$

Ex. Find a formula for the nth term of:

a. $\left\{-\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, -\frac{6}{7}, \frac{7}{8}, \dots\right\}$

b. $\{-2, 3, -2, 3, -2, 3, \dots\}$

a. $a_n = (-1)^n \frac{n+1}{n+2}$

b. $a_{2n-1} = -2$
$a_{2n} = 3$

Some sequences don't have an easy formula for the nth term:

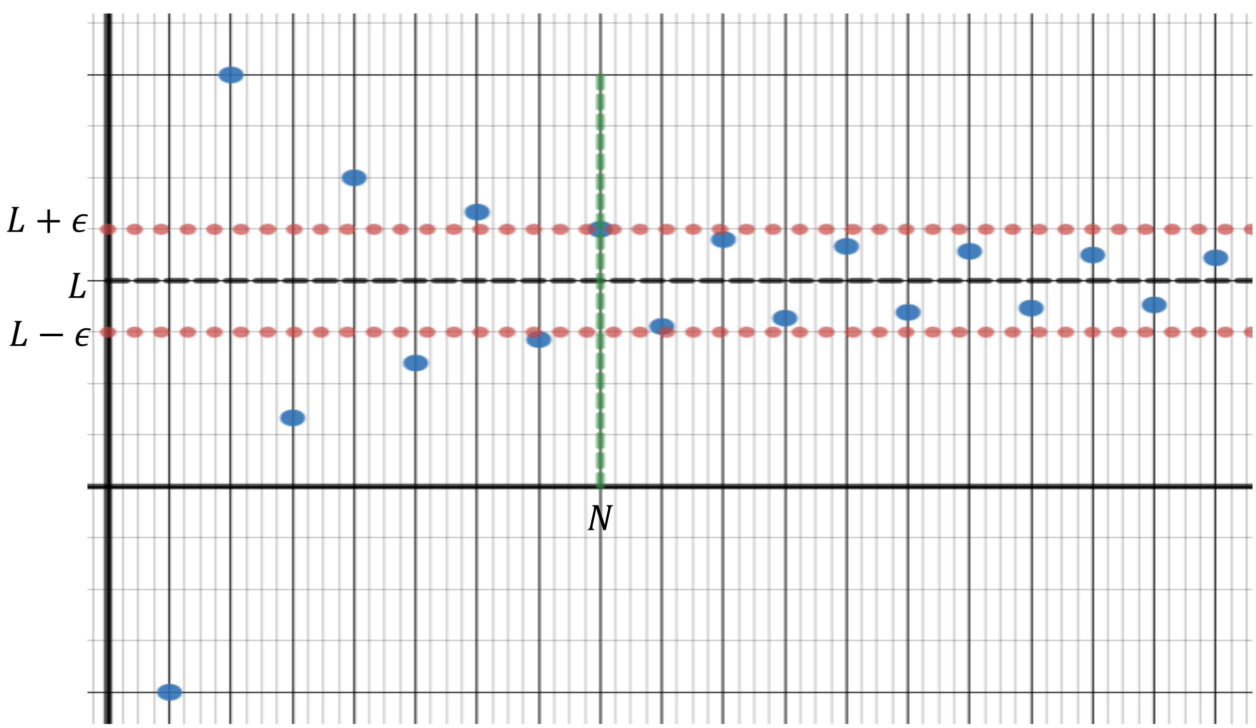
Ex. The Fibonacci sequence:

$$a_1 = 1, \quad a_2 = 1, \quad a_3 = 2, \quad a_4 = 3, \quad a_5 = 5, \quad a_6 = 8, \quad a_7 = 13,$$

$$a_n = a_{n-1} + a_{n-2}$$

Given a sequence of numbers $\{a_n\}$, we can ask if the sequence **converges** to some number.

Def. A sequence $\{a_n\}$ has a **limit L**, written $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$, if for every $\epsilon > 0$ there is an integer N such that if $n > N$, then $|a_n - L| < \epsilon$. If $\lim_{n\to\infty} a_n = L < \infty$, then we say the sequence $\{a_n\}$ **converges**. If a sequence doesn't converge, then we say it **diverges**.



Ex. $\left\{\frac{1}{n}\right\} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$ converges to 0 since $\lim_{n\to\infty} \frac{1}{n} = 0.$

The limit properties we know for functions also hold for sequences.

Theorem: If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and C is a constant, then:

1. $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
3. $\lim_{n\to\infty}(Ca_n) = C\lim_{n\to\infty}(a_n)$
4. $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty}(a_n)\lim_{n\to\infty}(b_n)$
5. $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty} a_n / \lim_{n\to\infty} b_n;$ if $\lim_{n\to\infty} b_n \neq 0.$
6. $\lim_{n\to\infty}(a_n)^p = (\lim_{n\to\infty} a_n)^p;$ if $p > 0 \; and \;\; a_n > 0$

The Squeeze Theorem also has a version for sequences.

Squeeze Theorem for functions: If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$ and

$f(x) \le h(x) \le g(x)$, then $\lim_{x\to a} h(x) = L$.

Squeeze theorem for Sequences: If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Ex. Find $\lim_{n\to\infty} \left|\frac{\sin(n)}{n}\right|$.

Notice that $0 \le |\sin(x)| \le 1$ for all x That means:

$$\frac{0}{n} \le \left|\frac{\sin(n)}{n}\right| \le \frac{1}{n} \quad \text{or} \quad 0 \le \left|\frac{\sin(n)}{n}\right| \le \frac{1}{n}.$$

Since $\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, by the squeeze theorem $\lim_{n\to\infty} \left|\frac{\sin(n)}{n}\right| = 0.$

Here's another useful theorem:

Theorem: If $\lim_{n\to\infty} |\, a_n \,| = 0$, then $\lim_{n\to\infty} a_n = 0.$

Ex. We saw that $\lim_{n\to\infty} \left|\frac{\sin(n)}{n}\right| = 0$. Then by this theorem $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0.$

Theorem: If $\lim_{x\to\infty} f(x) = l$ and $f(n) = a_n$ then $\lim_{n\to\infty} a_n = l.$

Ex. Find the limit of the following sequences if they exist:

a. $a_n = \frac{n^2+n}{2n^2+1}$

b. $a_n = \frac{\ln(n)}{n}$

c. $a_n = \frac{(-1)^n}{n}$

d. $a_n = \frac{2^n}{n^2}$

e. $a_n = \frac{n!}{n^n}$

f. $a_n = (-1)^n$

a. $\lim_{x\to\infty} \frac{x^2+x}{2x^2+1} = \lim_{x\to\infty} \frac{2x+1}{4x} = \lim_{x\to\infty} \frac{2}{4} = \frac{1}{2}$ by L'Hospital's Rule

so: $\lim_{n\to\infty} \frac{n^2+n}{2n^2+1} = \frac{1}{2}$.

b. $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{1} = 0$ by L'Hospital's Rule

so: $\lim_{n\to\infty} \frac{\ln n}{n} = 0.$

c. $\left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$, and

$\lim_{x\to\infty} \frac{1}{x} = 0$

so: $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$ Here we are using the previous 2 theorems.

d. $\lim_{x\to\infty} \frac{2^x}{x^2} = \lim_{x\to\infty} \frac{(\ln 2)2^x}{2x} = \lim_{x\to\infty} \frac{(\ln 2)^2 2^x}{2} = \infty$, by L'Hospital's Rule.

So: $\lim_{n\to\infty} \frac{2^n}{n^2} = \infty$, ie, $\{\frac{2^n}{n^2}\}$ diverges.

e. $0 \le a_n = \frac{1\cdot 2\cdot 3\cdot \ldots \cdot n}{n\cdot n\cdot \ldots \cdot n} = \frac{1}{n}\left(\frac{2}{n}\cdot\frac{3}{n}\cdot \ldots \cdot\frac{n}{n}\right) \le \frac{1}{n}$ so by the Squeeze Theorem, we know $0 \le a_n \le \frac{1}{n}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$

so: $\lim_{n\to\infty} a_n = 0.$

f. $a_n = (-1)^n$ thus $\{a_n\} = -1, 1, -1, 1, -1, 1, \ldots$ so $\{a_n\}$ diverges.

Ex. $a_n = r^n$; for what values of r is the sequence convergent?

From the graph of exponential functions we know:

$$\lim_{x \to \infty} a^x = \begin{cases} \infty & \text{if } a > 1 \\ 0 & \text{if } 0 < a < 1 \end{cases}$$

so

$$\lim_{n \to \infty} r^n = \begin{cases} \infty & \text{if } r > 1 \\ 0 & \text{if } 0 < r < 1 \end{cases}$$

If $r = 1$, then $\lim_{n \to \infty} 1^n = 1$

If $r = 0$, then $\lim_{n \to \infty} 0^n = 0$

If $-1 < r < 0$, then $0 < |r| < 1$ so:

$$\lim_{n \to \infty} |r|^n = 0 \;\; \Rightarrow \;\; \lim_{n \to \infty} r^n = 0.$$

If $r \le -1$, then $\{r^n\}$ diverges because it oscillates

similarly to $a_n = (-1)^n$.

Ex. Determine the convergence of $a_n = \frac{3^{n+1}}{4^n}$.

$$a_n = \frac{3^n \cdot 3}{4^n} = 3 \cdot \left(\frac{3}{4}\right)^n$$

$b_n = \left(\frac{3}{4}\right)^n$ converges to $0 \Rightarrow a_n = 3\left(\frac{3}{4}\right)^n$ converges to 0 too.

Ex. Determine the convergence of $a_n = \frac{2(n+1)!}{n!}$.

$$a_n = \frac{2(n+1)(n!)}{n!} = 2(n+1) \text{ diverges to } \infty.$$

Def. A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is:

$$a_1 < a_2 < a_3 < a_4 < \cdots < a_n < \cdots$$

It's called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$, that is:

$$a_1 > a_2 > a_3 > a_4 > \cdots > a_n > \cdots$$

Ex. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots, \frac{1}{n}, \ldots$ is a decreasing sequence because:

$$a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$$

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots, \frac{n}{n+1}, \ldots$ is an increasing sequence because

$$a_{n+1} = \frac{n+1}{n+2} \text{ and } a_n = \frac{n}{n+1}$$

so $\frac{n+1}{n+2} > \frac{n}{n+1}$ exactly when $(n+1)^2 > n(n+2)$.

Multiplying out both sides we get:

$$n^2 + 2n + 1 > n^2 + 2n$$

This is equivalent to $1 > 0$, which of course is true, so:

$a_n < a_{n+1}$ for all $n \geq 1$.

Another handy way to show that a sequence is increasing or decreasing is to show that the associated function (what you get when you put "x" in for "n") is either increasing or decreasing for $x \geq 1$.

Ex. Show $a_n = \frac{n}{n+1}$ is increasing.

The associated function is $f(x) = \frac{x}{x+1}$. Using the quotient rule we have:

$$f'(x) = \frac{1}{(x+1)^2} > 0, \text{ for } x \geq 1.$$

So, $f(x)$ is increasing for $n \geq 1$, hence $a_n = \frac{n}{n+1}$ is increasing.

Def. A sequence $\{a_n\}$ is **bounded above** if there is some number M such that

$$a_n \leq M, \text{ for all } n \geq 1.$$

A sequence is **bounded below** if there is a number m such that $m \leq a_n$, for all $n \geq 1$.

A sequence is **bounded** if it is bounded above and bounded below.

Ex. $\{a_n\} = \{n\}$ is bounded below by 1, but not bounded above (and thus not a bounded sequence).

Ex. $\{a_n\} = \{\frac{1}{n}\}$ is bounded below by 0 and bounded above by 1, thus bounded.

Ex. $\{a_n\} = \left\{\frac{\cos(n\pi)}{n}\right\} = -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \ldots$ is bounded above by $\frac{1}{2}$ and below by -1, thus bounded.