Hooke's Law

In physics we learn that the work, W, to move a mass, m, a distance of d, with a constant force, F, along a line is given by:

$$W = Fd\ .$$

Ex. Find the work done to lift a 20 lb bag of groceries 3 feet.

$$W = Fd = (20)(3) = 60 \text{ foot-pounds.}$$

In the metric system, we would measure the distance moved in meters rather than feet and the force in Newtons (one Newton is the force on a mass of one kilogram to produce an acceleration of one meter per second squared), rather than pounds. Thus, in the metric system we measure the work done in Newton-meters, called joules, instead of foot-pounds .

$$1 \text{ pound} = \frac{1}{2.2}kg(9.8)\frac{m}{s^2} \approx 4.455\frac{kg-m}{s^2} = 4.455N$$

$$1 \text{ foot} \approx .305m$$

$$1 \text{ foot-pound} \approx (4.455)(.305) \approx 1.36N\text{-}m = 1.36J\ .$$

So in our previous example:

$$60 \text{ foot-pounds} = (60)(1.36)\,J = 81.6\,J\ .$$

But how do we calculate the work done when the force is variable? For example, suppose a spring at rest is attached to a wall. If we pull on the unattached end of the spring, the further we stretch it the harder it becomes to stretch. Thus, the force exerted on the spring to stretch it is variable. This is expressed by Hooke's Law:

$$F = kx$$

where F is the force, x is the distance the spring has been stretched (or compressed) from rest, and k is the spring constant.

Ex. A force of $90\ N$ is required to maintain a spring stretched $\frac{1}{2}m$ from its rest length. Find the spring constant for this spring.

$$F = kx$$

$$90 = k\left(\frac{1}{2}\right)$$

$$k = 180\frac{N}{m}.$$

We can generalize the case of constant force to a variable force by saying the force is approximately constant over any small interval Δx. Suppose we want to find the work done to move and object from $x = a$ to $x = b$ if the force is variable and given by $f(x)$.

Let's break up the interval $[a, b]$ into a collection of n equal sub-intervals with endpoints where $a = x_0, x_1, x_2, \dots, x_n = b$, with $x_i - x_{i-1} = \Delta x$ for all $i = 1, \dots, n$. On any sub-interval $[x_{i-1}, x_i]$, the force is approximately constant at $f(x_i)$, so the work done on that interval is approximately: $f(x_i)\Delta x$.

Adding up the work on all of the sub-intervals we get:

$$W \approx \sum_{i=1}^{n} f(x_i)\,\Delta x\ .$$

If we take the limit as Δx goes to zero we get:

$$W = \int_{x=a}^{x=b} f(x)dx \,.$$

Ex. Find the work done to stretch a spring from rest $\frac{1}{2}$ meter if the force is given by $F = 200x$, where F is in Newtons.

$$W = \int_0^{\frac{1}{2}} 200x\,dx = 100x^2\Big|_0^{\frac{1}{2}} = 100\left(\frac{1}{4}\right) - 100(0) = 25\,J\,.$$

Ex. Using the spring in the previous example, find the work done to stretch the spring from $\frac{1}{2}$ meter to 1 meter.

$$W = \int_{\frac{1}{2}}^{1} 200x\,dx = 100x^2\Big|_{\frac{1}{2}}^{1} = 100(1)^2 - 100\left(\frac{1}{2}\right)^2$$

$$= 100 - 25 = 75\,J\,.$$

Ex. The force required to hold a spring stretched 6 inches from its resting position is 30 pounds. Find the work required to stretch the spring from 3 inches to 9 inches.

First we have to find the spring constant.

Here work is expressed in foot-pounds.

$$F = kx$$

$$30 = k\left(\frac{1}{2}\right) \; ; \quad 6 \text{ inches} = \frac{1}{2} \text{ foot}$$

$$k = 60$$

$$W = \int_{\frac{1}{4}}^{\frac{3}{4}} 60x \, dx = 30x^2 \Big|_{\frac{1}{4}}^{\frac{3}{4}} = 30\left[\left(\frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2\right]$$

$$= 30\left[\frac{9}{16} - \frac{1}{16}\right] = 15 \text{ foot-pounds.}$$

Ex. $3\ J$ of work are needed to stretch a spring from a rest length of 25 cm to 50 cm.

a) How much work is necessary to stretch the spring from a length of 45 cm to 50 cm?
b) How much beyond its rest length will the spring be maintained with a force of $72\ N$?

a) $W = \int_a^b f(x)dx$; where $f(x) = kx$.

We know that it take $3\ J$ of work to stretch the spring $.25$ m (50 cm – 25 cm $=$ 25 cm $= .25$ m) from rest. So we have:

$$3 = \int_0^{.25} kxdx = \left.\frac{kx^2}{2}\right|_0^{.25} = \frac{k}{2}((.25)^2 - 0^2) = \frac{k}{2}(.0625)$$

$$\Rightarrow k = 96\,.$$

Work done to stretch the spring from 45 cm ($x = 45$ cm – 25 cm $= .2\ m$) to 50 cm ($x = 50$ cm – 25 cm $= .25\ m$):

$$W = \int_{.2}^{.25} 96x\ dx = \left.\frac{96x^2}{2}\right|_{.2}^{.25} = 48[(.25)^2 - (.2)^2]$$

$$= 48[(.0625) - (.04)] = 1.08\,J.$$

b)

$$F = 96x$$

$$72 = 96x$$

$$\frac{3}{4} = x$$

So a force of $72\ N$ will maintain the spring $.75$ meters $= 75$ cm beyond its rest length.