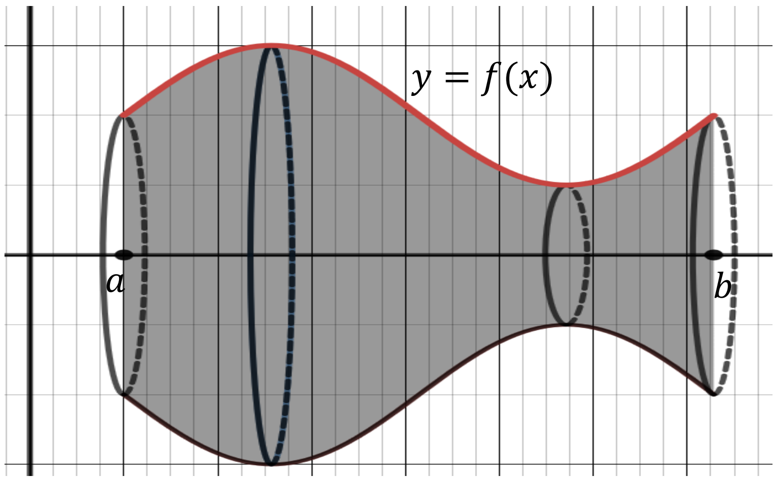
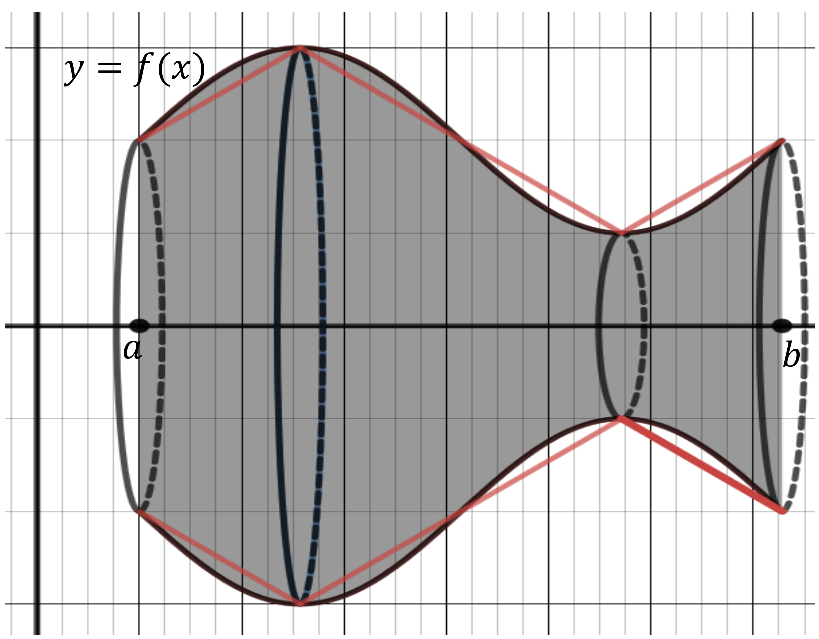
Area of a Surface of Revolution

Def. If the graph of a continuous function is revolved about a line, then the resulting surface is called a **surface of revolution**.

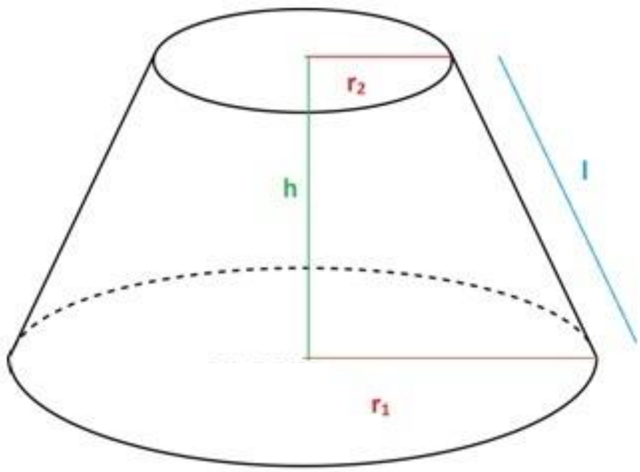


Our goal is to find a formula for the surface area of a surface of revolution. We will start with a curve $y = f(x)$ and revolve it about the x-axis. Notice that if we have a line segment (that is neither parallel nor perpendicular to the x-axis) and revolve it about the x-axis we get a portion of a cone called a frustum.

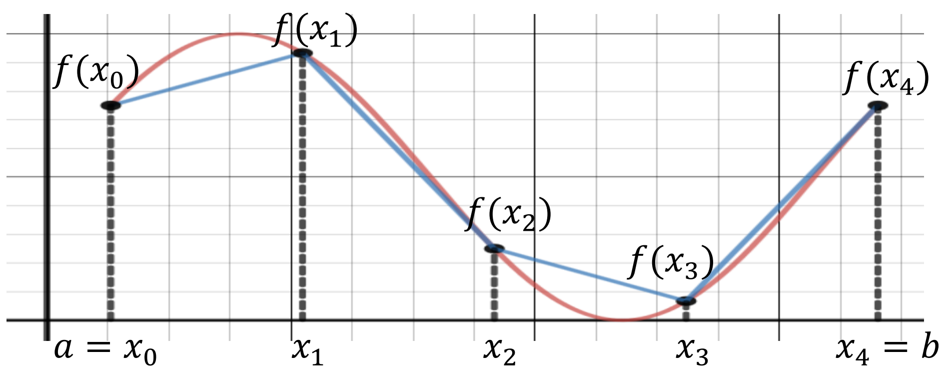


Using the formulas for the area of a cone ($A = \pi r l,\ l =$slant height) and similar triangles, one can derive the formula for the surface area of a frustum:

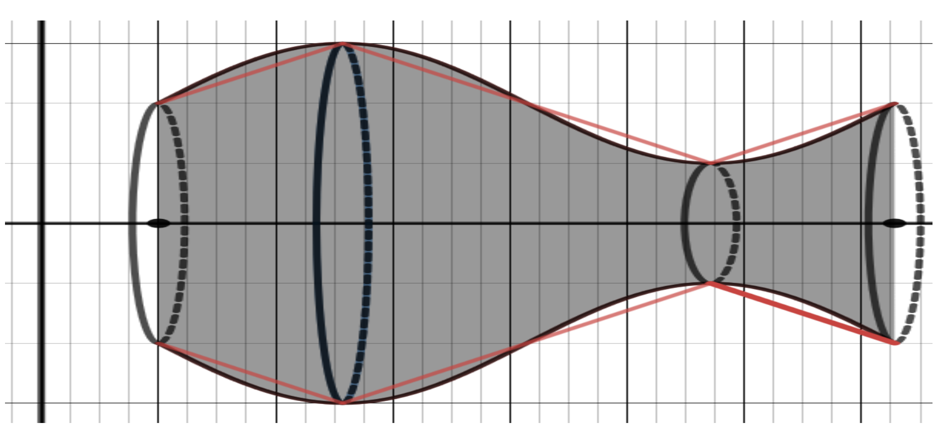
$$A = 2\pi r l\,; \qquad r = \frac{r_1+r_2}{2}\,; \qquad l = \text{slant height of frustum}$$



When we found the formula for the length of a curve we approximated the curve with line segments.



We will do that same thing with a curve generating a surface of revolution. Notice when each line segment is revolved about the x-axis it creates a frustum.



The area of the frustum created from revolving the line segments between (x_{i-1}, y_{i-1}) and (x_i, y_i) is written as:

$$\Delta A_i = 2\pi \left(\frac{y_{i-1}+y_i}{2}\right)\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

Just like we did when finding the length of a curve:

$$\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + \left(f'(x_i^*)(\Delta x)\right)^2}$$

$$= \sqrt{1 + \left(f'(x_i^*)\right)^2}\ \Delta x$$

So we can write:

$$\Delta A_i = 2\pi \left(\frac{y_{i-1}+y_i}{2}\right)\sqrt{1 + \left(f'(x_i^*)\right)^2}\ \Delta x$$

$$\text{Surface Area} = \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi\ f(x_i^*)\sqrt{1 + \left(f'(x_i^*)\right)^2}\ \Delta x$$

$$\textbf{Surface Area} = \int_a^b 2\pi\, f(x)\sqrt{1 + \left(f'(x)\right)^2}\, dx$$

OR

$$\textbf{Surface Area} = \int_a^b 2\pi\, y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

If the curve is given as $x = g(y)\ ;\ \ c \le y \le d$, then we have:

$$\textbf{Surface Area} = \int_c^d 2\pi\, y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

By using the following fact (where s is the arc length function):

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

We can now write:

$$S = \int 2\pi\, y\, ds$$

If we are revolving a curve about the y-axis, then we have:

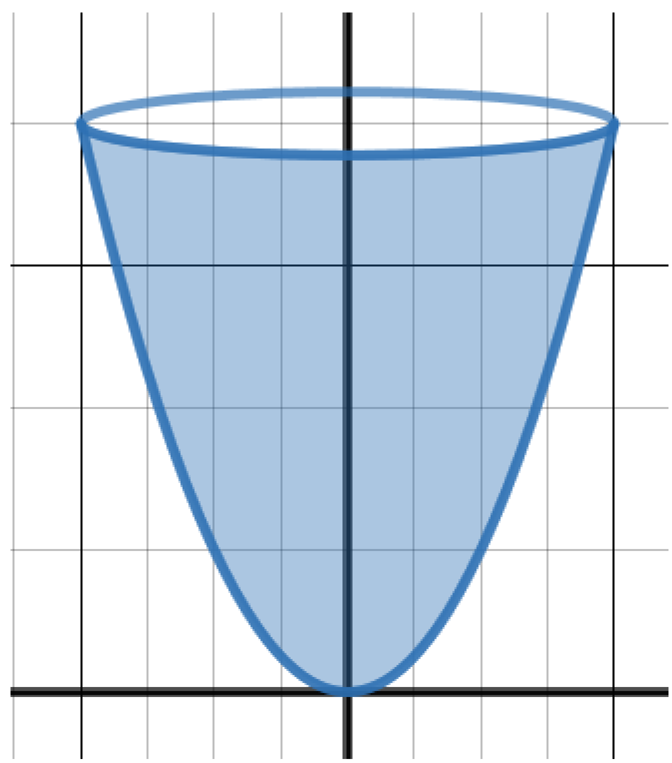
$$\textbf{S.A.} = \int_{x=a}^{x=b} 2\pi\, x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$y = f(x)\ ;\ \ a \le x \le b$

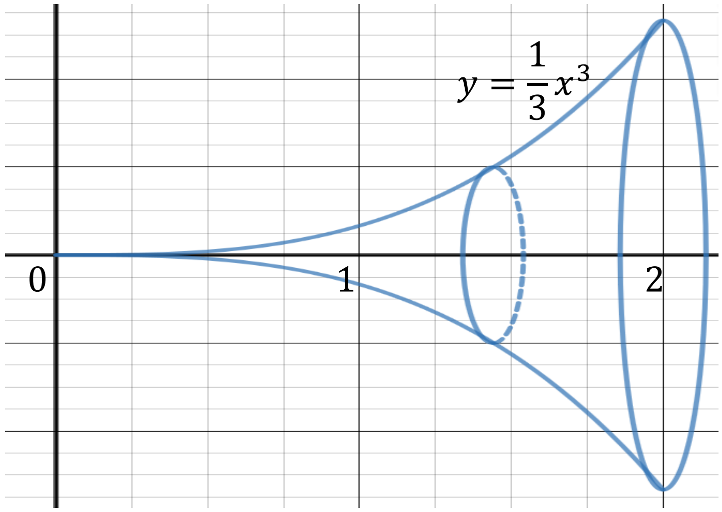
OR

$$\textbf{S.A.} = \int_{y=c}^{y=d} 2\pi\, x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$x = g(y)\ ;\ \ c \le y \le d$



Ex. If $y = \frac{1}{3}x^3\ , 0 \le x \le 2$ is rotated about the x- axis, then find the surface area of the resulting surface.



$$y = \frac{1}{3}x^3$$

$$\frac{dy}{dx} = x^2$$

$$\text{S. A.} = \int_{x=a}^{x=b} 2\pi\, y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{x=0}^{x=2} 2\pi\, \left(\frac{1}{3}x^3\right) \sqrt{1 + (x^2)^2}\, dx$$

$$= \frac{2\pi}{3} \int_0^2 x^3 (1 + x^4)^{\frac{1}{2}}\, dx$$

Let $u = 1 + x^4 \quad \Rightarrow \quad x = 0,\ u = 1; \qquad x = 2,\ u = 17.$

$du = 4x^3 dx$

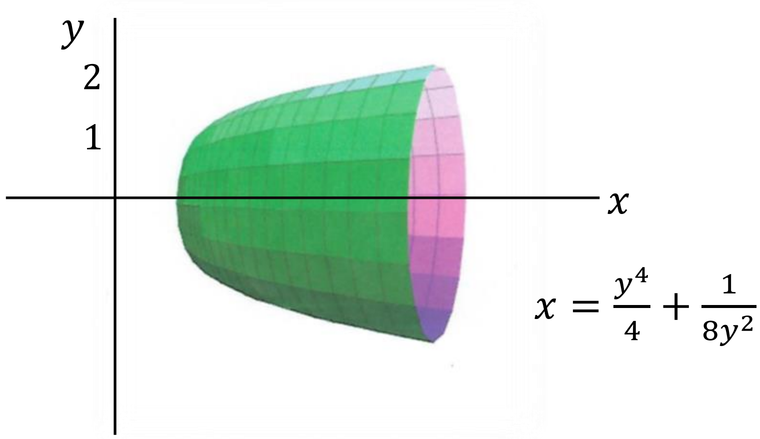
$\frac{1}{4} du = x^3 dx$

$$= \frac{2\pi}{3} \int_{u=1}^{u=17} u^{\frac{1}{2}} \left(\frac{1}{4}\right)\, du$$

$$= \frac{\pi}{6} \left(\frac{2}{3} u^{\frac{3}{2}}\right) \Big|_{u=1}^{u=17}$$

$$= \frac{\pi}{9} \left(17^{\frac{3}{2}} - 1\right)$$

Ex. The curve $x = \frac{y^4}{4} + \frac{1}{8y^2}$ from $y = 1$ to $y = 2$ is rotated about the x-axis. Find the surface area generated.



$$\text{S.A.} = \int_{y=c}^{y=d} 2\pi\, y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$\frac{dx}{dy} = y^3 - \frac{1}{4y^3}$$

$$\text{S.A.} = \int_{y=1}^{y=2} 2\pi\, y \sqrt{1 + \left(y^3 - \frac{1}{4y^3}\right)^2}\, dy$$

$$= 2\pi \int_{y=1}^{y=2} y \left(1 + \left(y^6 - \frac{1}{2} + \frac{1}{16y^3}\right)\right)^{\frac{1}{2}} dy$$

$$= 2\pi \int_{y=1}^{y=2} y \left(y^6 + \frac{1}{2} + \frac{1}{16y^3}\right)^{\frac{1}{2}} dy$$

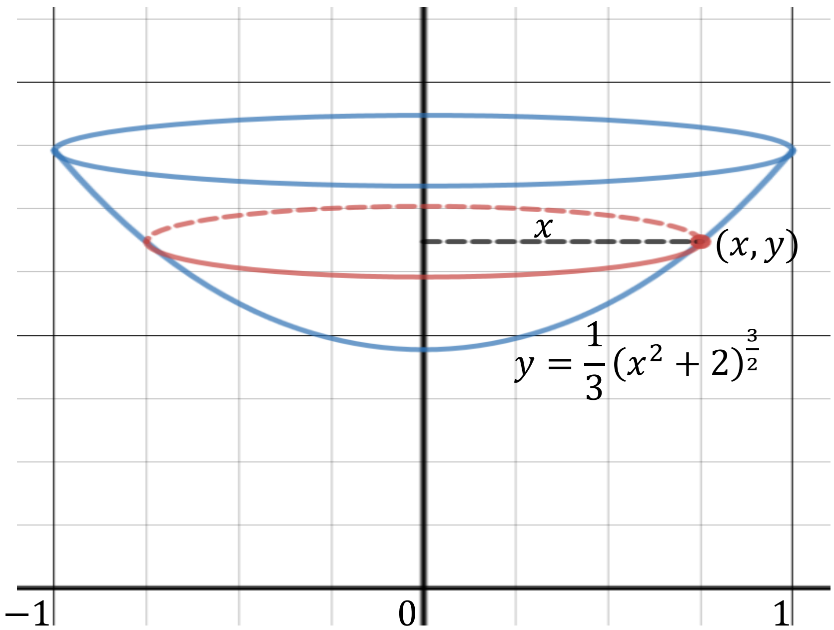
$$= 2\pi \int_{y=1}^{y=2} y \left(\left(y^3 + \frac{1}{4y^3}\right)^2\right)^{\frac{1}{2}} dy \;=\; 2\pi \int_{y=1}^{y=2} y \left(y^3 + \frac{1}{4y^3}\right) dy$$

$$= 2\pi \int_{y=1}^{y=2} y^4 + \frac{1}{4} y^{-2} dy \;=\; 2\,\pi \left(\frac{y^5}{5} - \frac{1}{4} y^{-1}\right)\Big|_{y=1}^{y=2}$$

$$= 2\pi \left[\left(\frac{2^5}{5} - \frac{1}{8}\right) - \left(\frac{1}{5} - \frac{1}{4}\right)\right] = \;2\pi \left(\frac{31}{5} + \frac{1}{8}\right)$$

$$= \frac{253\pi}{20}.$$

Ex. The curve $y = \frac{1}{3}(x^2+2)^{\frac{3}{2}}$, $0 \le x \le 1$, is revolved about the y-axis.
Find the surface area generated.



$$\text{S.A.} = \int_{x=a}^{x=b} 2\pi\, x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$y = \frac{1}{3}(x^2+2)^{\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(x^2+2)^{\frac{1}{2}}(2x)$$

$$= x(x^2+2)^{\frac{1}{2}}$$

$$\text{S.A.} = \int_{x=0}^{x=1} 2\pi\, x \sqrt{1+\left[x(x^2+2)^{\frac{1}{2}}\right]^2}\, dx$$

$$= \int_{x=0}^{x=1} 2\pi\, x \sqrt{1+x^2(x^2+2)}\,dx$$

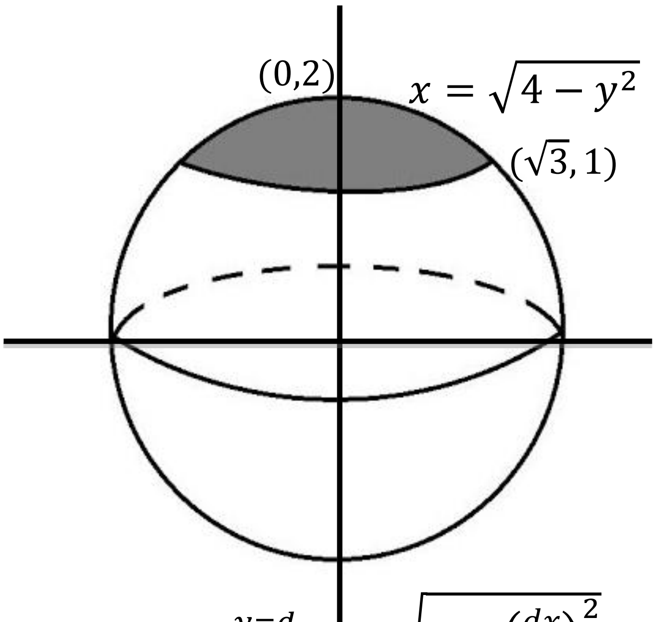
$$= 2\pi \int_0^1 x\sqrt{1+x^4+2x^2}\,dx \;=\; 2\pi \int_0^1 x\sqrt{(x^2+1)^2}\,dx$$

$$= 2\pi \int_0^1 x\,(x^2+1)\,dx \;=\; 2\pi \int_0^1 (x^3+x)\,dx$$

$$= 2\pi\left(\frac{x^4}{4}+\frac{x^2}{2}\right)\Big|_0^1 \;=\; 2\pi\left(\frac{1}{4}+\frac{1}{2}\right)$$

$$= \frac{3\pi}{2}.$$

Ex. The curve $x = \sqrt{4-y^2}$, $1 \le y \le 2$, is part of the circle of radius 2 given by $x^2 + y^2 = 4$. Find the area of the surface generated by revolving this curve about the y-axis.



$$\text{S.A.} = \int_{y=c}^{y=d} 2\pi\, x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$x = \sqrt{4-y^2} = (4-y^2)^{\frac{1}{2}}$$

$$\frac{dx}{dy} = \frac{1}{2}(4-y^2)^{-\frac{1}{2}}(-2y) = -\frac{y}{\sqrt{4-y^2}}$$

$$\text{S.A.} = \int_{y=1}^{y=2} 2\pi \sqrt{4-y^2} \sqrt{1 + \frac{y^2}{4-y^2}}\, dy$$

$$= 2\pi \int_1^2 \sqrt{4-y^2} \sqrt{\frac{4-y^2}{4-y^2} + \frac{y^2}{4-y^2}}\, dy$$

$$= 2\pi \int_1^2 \sqrt{4-y^2} \sqrt{\frac{4}{4-y^2}}\, dy = 2\pi \int_1^2 \sqrt{4-y^2} \frac{\sqrt{4}}{\sqrt{4-y^2}}\, dy$$

$$= 2\pi \int_1^2 2\, dy = 4\pi y\Big|_1^2$$

$$= 4\pi(2-1) = 4\pi.$$