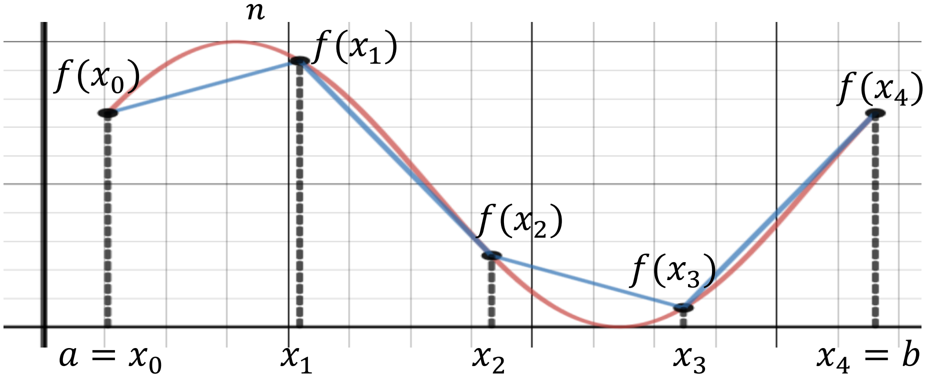
Arc Length

To find the length of a continuously differentiable curve $y = f(x);\ a \le x \le b$, we divide the interval $[a, b]$ into n equal subintervals. The length of each subinterval is $\Delta x = \frac{b-a}{n}$.



We then approximate the length of the curve for $x_{i-1} \le x \le x_i$ with the length of the line segment connecting $(x_{i-1}, y_{i-1}), (x_i, y_i)$. That length is written as:

$$\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

By the mean value theorem there is a $c_i = x_i^* \in [x_{i-1}, x_i]$ such that:

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})\,; \qquad x_i^* \in [x_{i-1}, x_i]$$

$$\Delta y_i = f'(x_i^*)\Delta x$$

If we add up the lengths of all of these line segments and let the number of subintervals, n, go to infinity, then we get:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(f'(x_i^*)\right)^2}\,\Delta x = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\,dx$$

or equivalently:

$$\boldsymbol{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.}$$

Ex. Find the length of the curve given by $y = \frac{2}{3}x^{\frac{3}{2}} + 1$ for $0 \leq x \leq 3$.

$$L = \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$y = \frac{2}{3}x^{\frac{3}{2}} + 1$$

$$\frac{dy}{dx} = x^{\frac{1}{2}}$$

$$\left(\frac{dy}{dx}\right)^2 = x$$

$$L = \int_0^3 \sqrt{1 + x}\, dx = \frac{2}{3}(1 + x)^{\frac{3}{2}}\Big|_{x=0}^{x=3}$$

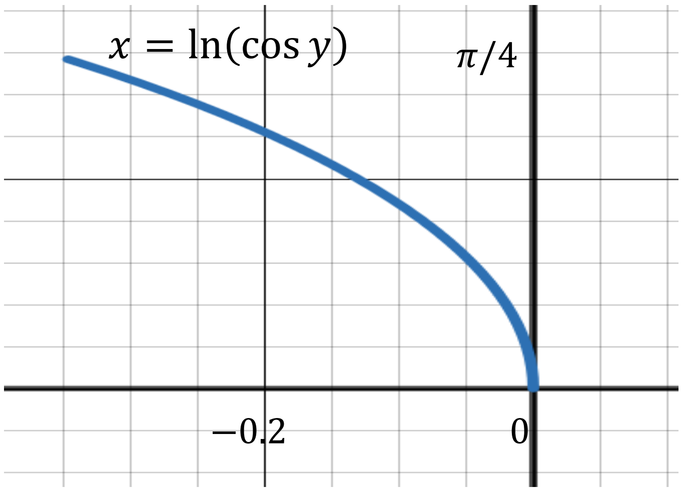
$$= \frac{2}{3}\left[(1+3)^{\frac{3}{2}} - (1+0)^{\frac{3}{2}}\right] = \frac{2}{3}\left[4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}[8 - 1] = \frac{14}{3}.$$

If a curve is given by $x = g(y),\ c \leq y \leq d$ where $g'(y)$ is continuous, then a similar argument to the case where $y = f(x)$ gives us:

$$\boldsymbol{L = \int_{y=c}^{y=d} \sqrt{1 + \left(g'(y)\right)^2}\, dy = \int_{y=c}^{y=d} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.}$$

Ex. Find the length of $x = \ln(\cos y)$; $0 \le y \le \frac{\pi}{4}$.



$$\frac{dx}{dy} = \frac{1}{\cos y}(-\sin y) = -\tan y$$

$$\left(\frac{dx}{dy}\right)^2 = \tan^2 y$$

$$L = \int_{y=0}^{y=\frac{\pi}{4}} \sqrt{1 + \tan^2 y}\ dy$$

$$= \int_{y=0}^{y=\frac{\pi}{4}} \sqrt{\sec^2 y}\ dy = \int_{y=0}^{y=\frac{\pi}{4}} \sec y\ dy$$

We saw earlier that:

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C$$

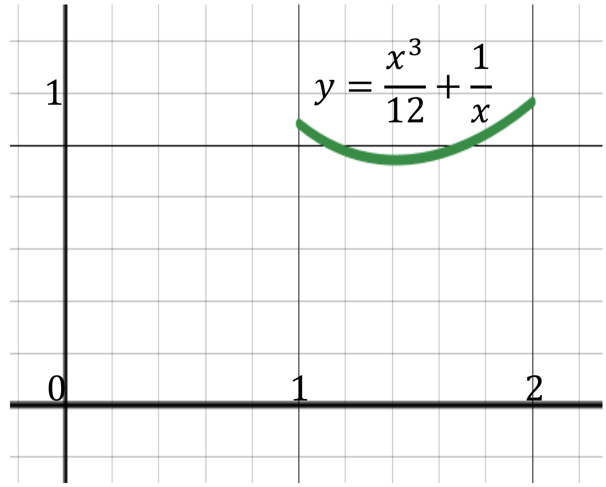
$$\Longrightarrow \quad L = \ln|\sec y + \tan y|\Big|_{y=0}^{y=\frac{\pi}{4}}$$

$$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0|$$

$$= \ln\left|\sqrt{2} + 1\right| - \ln 1$$

$$= \ln\left(\sqrt{2} + 1\right).$$

Ex. Find the length of the curve given by $y = \frac{x^3}{12} + \frac{1}{x}$; $\quad 1 \le x \le 2$



$$\frac{dy}{dx} = \frac{x^2}{4} - \frac{1}{x^2}$$

$$L = \int_1^2 \sqrt{1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{1 + \left(\frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}\right)}\, dx$$

$$= \int_1^2 \sqrt{\left(\frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4}\right)}\, dx$$

$$= \int_1^2 \sqrt{\left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2}\, dx$$

$$= \int_1^2 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx$$

$$= \left(\frac{x^3}{12} - \frac{1}{x}\right)\Big|_{x=1}^{x=2}$$

$$= \left(\frac{8}{12} - \frac{1}{2}\right) - \left(\frac{1}{12} - 1\right)$$

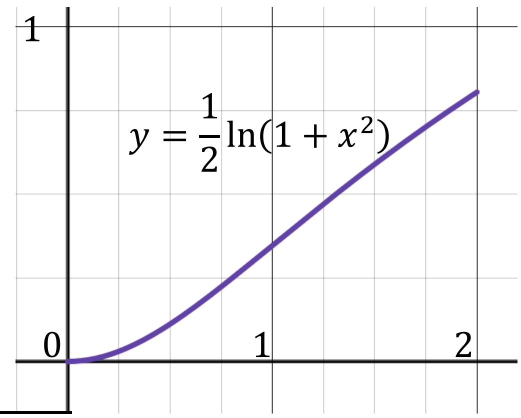
$$= \frac{13}{12}.$$

It's easy to find arc length problems where it's very difficult (or not possible) to find an elementary anti-derivative for the resulting integrand.

Ex. $y = \frac{1}{2}\ln(1 + x^2)$.

a) Set up the integral of the arc length of y for $0 \le x \le 2$

b) Use Simpson's Rule with $n = 4$ to approximate the value of the integral



a)
$$y = \frac{1}{2}\ln(1 + x^2)$$
$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{1+x^2}\right)(2x) = \frac{x}{1+x^2}$$
$$\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{(1+x^2)^2}$$

$$L = \int_{x=0}^{x=2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{x=0}^{x=2} \sqrt{1 + \frac{x^2}{(1 + x^2)^2}}\, dx$$

b) $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = .5$; and $f(x) = \sqrt{1 + \frac{x^2}{(1+x^2)^2}}$.

$x_0 = 0, \quad x_1 = .5, \quad x_2 = 1, \quad x_3 = 1.5, \quad x_4 = 2$

$$L = \int_0^2 \sqrt{1 + \frac{x^2}{(1+x^2)^2}}\, dx \approx \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)]$$

$$= \frac{0.5}{3}\left[1 + 4\sqrt{1 + \frac{(0.5)^2}{(1+(0.5)^2)^2}} + 2\sqrt{1 + \frac{1}{4}} + 4\sqrt{1 + \frac{(1.5)^2}{(1+(1.5)^2)^2}} + \sqrt{1 + \frac{4}{25}}\right]$$

$$L = \int_0^2 \sqrt{1 + \frac{x^2}{(1 + x^2)^2}}\, dx \approx 2.3506.$$

In the study of curves it's useful to introduce the arc length function, $s(x)$. This function measures the length of the curve $y = f(t)$ from a fixed point, $t = a$, to a variable point, $t = x$.

$$s(x) = \int_{t=a}^{t=x} \sqrt{1 + \left(f'(t)\right)^2}\, dt.$$

Notice that by The Fundamental Theorem of Calculus:

$$\frac{ds}{dx} = \sqrt{1 + \left(f'(t)\right)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

OR

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

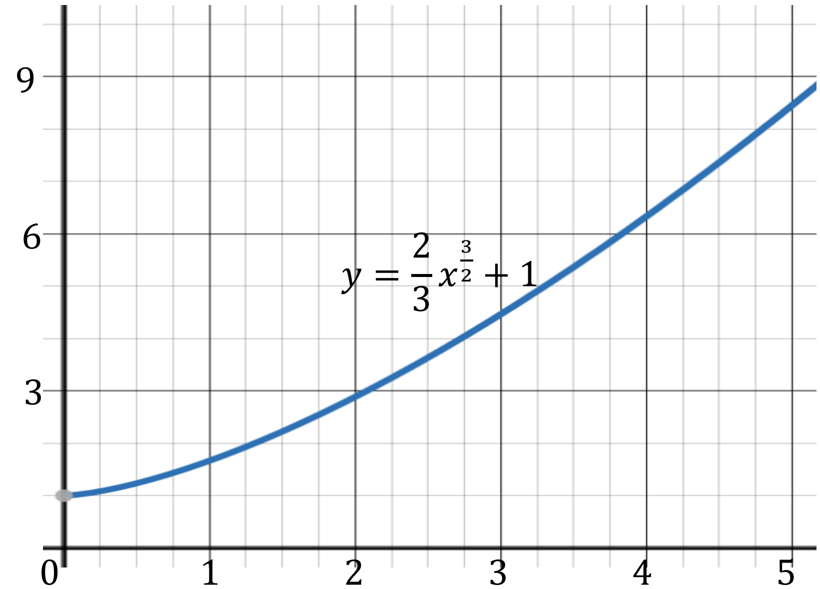
Also if $x = g(y)$, then we get:

$$\frac{ds}{dy} = \sqrt{1 + \left(g'(y)\right)^2} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

OR

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy\ .$$

Ex. Find the arc length function for the curve $y = \frac{2}{3}x^{\frac{3}{2}} + 1$ taking $(0, 1)$ as the starting point.



$$f(x) = \frac{2}{3}x^{\frac{3}{2}} + 1$$

$$f'(x) = x^{\frac{1}{2}}$$

$$\sqrt{1 + \left(f'(t)\right)^2} = \sqrt{1 + \left(t^{\frac{1}{2}}\right)^2} = \sqrt{1 + t}$$

$$s(x) = \int_0^x \sqrt{1 + \left(f'(t)\right)^2}\, dt$$

$$= \int_0^x \sqrt{1 + t}\, dt$$

$$= \frac{2}{3}(1 + t)^{\frac{3}{2}}\Big|_{t=0}^{t=x}$$

$$= \frac{2}{3}(1 + x)^{\frac{3}{2}} - \frac{2}{3}(1)^{\frac{3}{2}}$$

$$= \frac{2}{3}\left[(1 + x)^{\frac{3}{2}} - 1\right].$$

Notice that in the first example we calculated the length of $y = \frac{2}{3}x^{\frac{3}{2}} + 1$ for $0 \leq x \leq 3$, meaning we calculated $s(3)$ for the previous example.

$$s(3) = \frac{2}{3}\left[(1+3)^{\frac{3}{2}} - 1\right] = \frac{14}{3}$$

If we wanted the length of this curve between $x = 0$ and $x = 8$, then we would calculate $s(8)$.

$$s(8) = \frac{2}{3}\left[(1+8)^{\frac{3}{2}} - 1\right] = \frac{2}{3}\left(9^{\frac{3}{2}} - 1\right) = \frac{52}{3}.$$