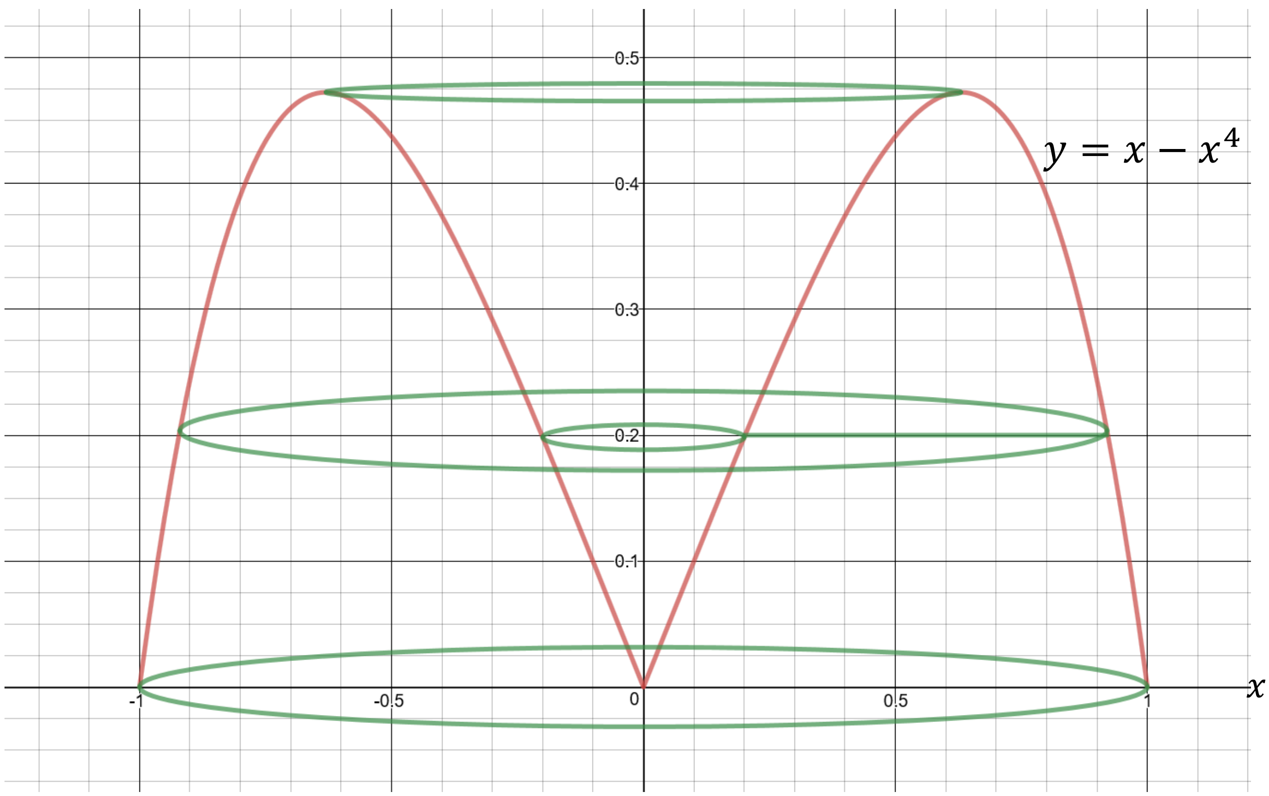
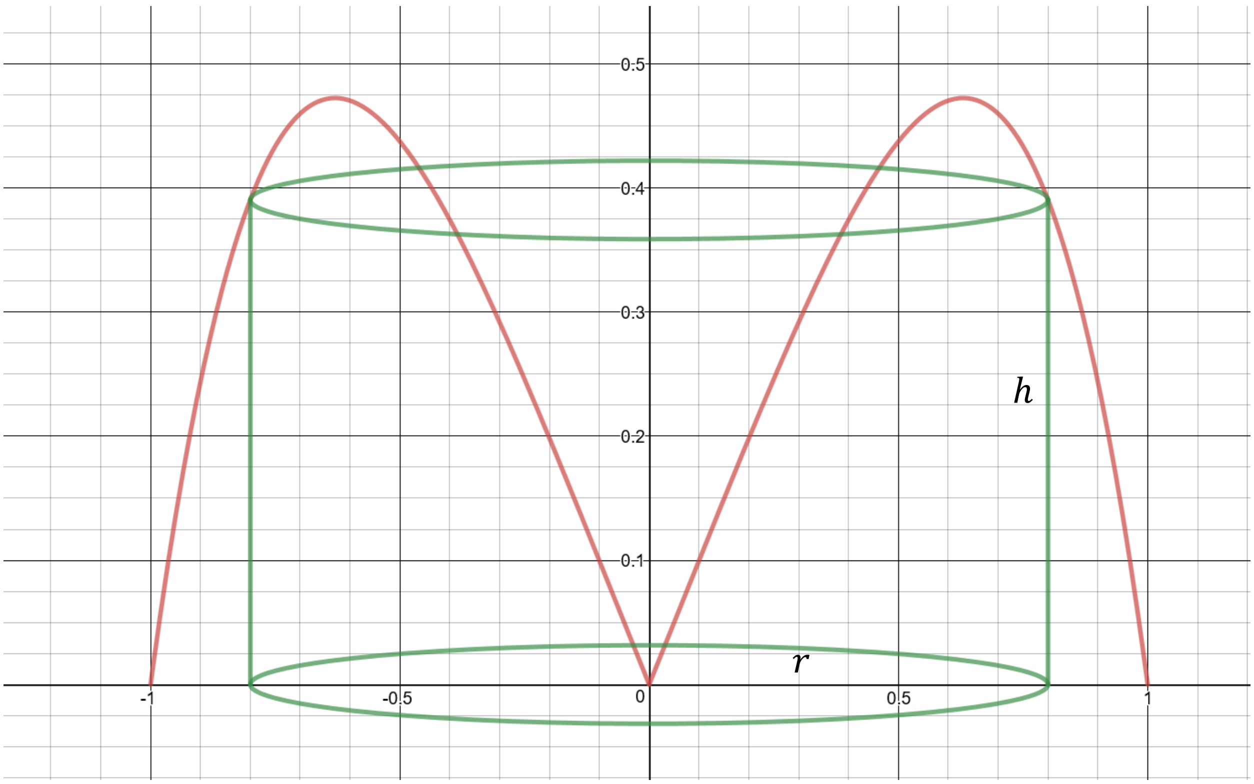
Volumes: Cylindrical Shells

Sometimes finding the volume of a solid revolution by slicing perpendicularly to a line of revolutions can lead to a very difficult (or impossible) problem. For example, if we rotate the region bounded by $y = x - x^4$ and the x-axis about the y-axis and slice it perpendicular to the y-axis, then we get an annulus. However, because this integral is done dy, we need to express the inner and outer radii in terms of y. This means we need to solve $y = x - x^4$ for x in terms of y (it can be done but it's very messy).



But what happens if we slice the region being rotated with a line parallel to the line of revolution and then rotate that line? The result will be a cylinder instead of a disk or annulus. So the “cross-sectional” area will be the lateral surface area (*not* including the disks on the top and the bottom of the cylinder). The formula for the lateral surface area of a cylinder is

$$A = 2\pi r h.$$



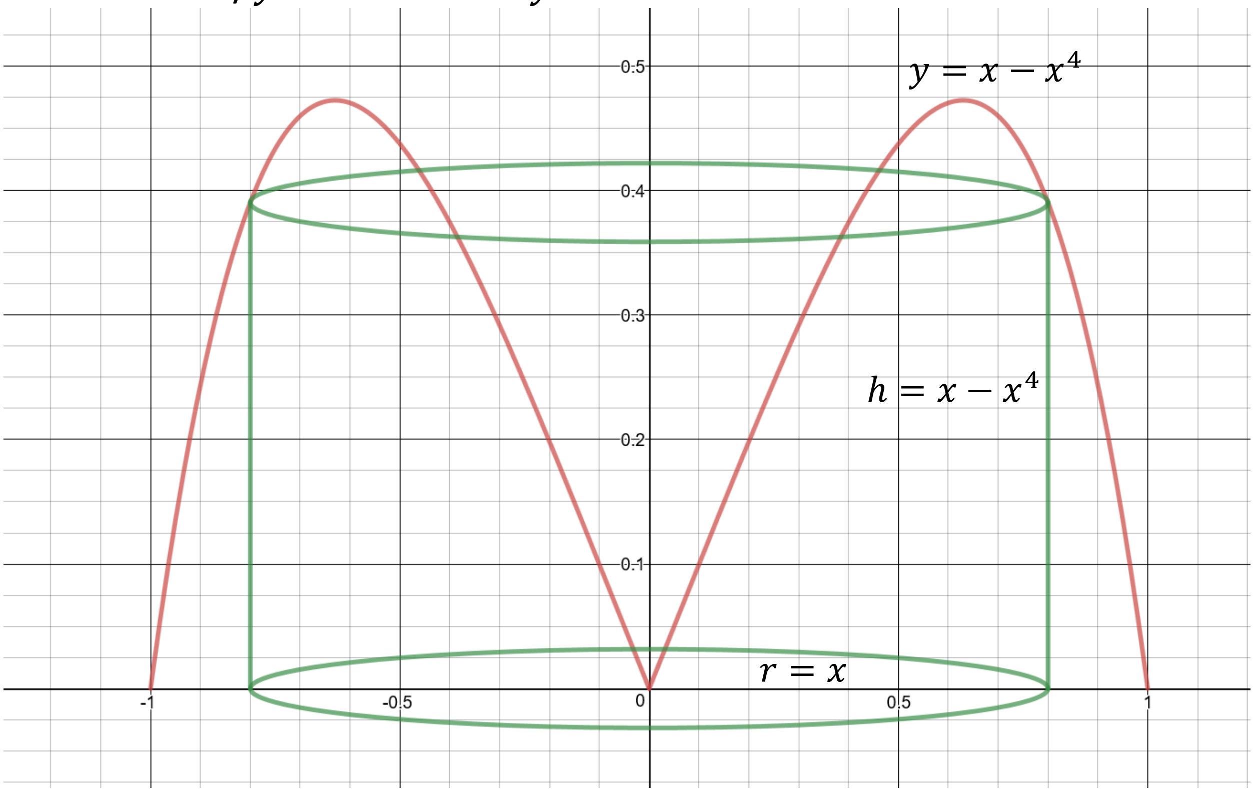
Thus, in each problem we have to find r and h in terms of x if we are slicing parallel to the y-axis and in terms of y if we are slicing parallel to the x-axis.

As before:

$$\boldsymbol{V = \int_{x=a}^{x=b} 2\pi r h\, dx \quad \text{or} \quad V = \int_{y=c}^{y=d} 2\pi r h\, dy}$$

This method is called the method of cylindrical shells.

Ex. Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = x - x^4$ and $y = 0$.



Since we are slicing parallel to the y-axis, the integral will be dx and we must find r and h in terms of x. In this case, $r = x$ and $h = y = x - x^4$. The curve $y = x - x^4$ intersects the x-axis when:

$$x - x^4 = 0$$
$$x(1 - x^3) = 0 \Rightarrow x = 0\,; x = 1$$

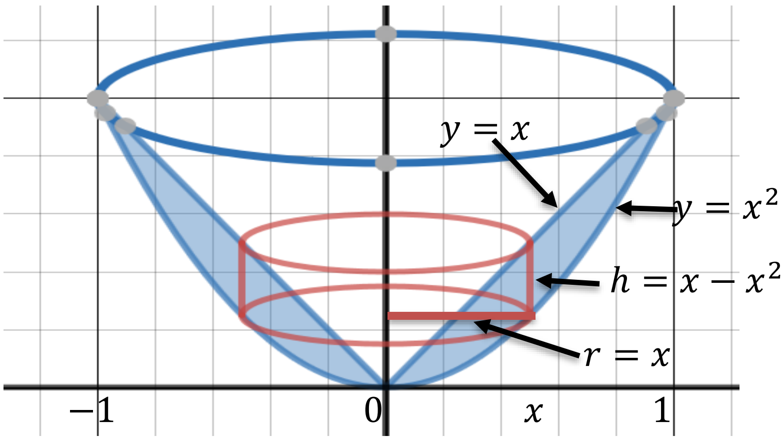
$$V = \int_{x=0}^{x=1} 2\pi r h \, dx = \int_{x=0}^{x=1} 2\pi x(x - x^4) \, dx$$

$$= 2\pi \int_{x=0}^{x=1} (x^2 - x^5) \, dx$$

$$= 2\,\pi \left(\frac{1}{3}x^3 - \frac{1}{6}x^6\right)\Big|_{x=0}^{x=1}$$

$$= 2\pi \left(\frac{1}{3} - \frac{1}{6}\right) = \frac{2\pi}{6} = \frac{\pi}{3}\,.$$

Ex. Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$. Use the method of cylindrical shells.



The curves $y = x$ and $y = x^2$ intersect when

$$x = x^2$$
$$0 = x^2 - x = x(x-1)$$
$$x = 0, 1.$$

Since the line of revolution is the y-axis, we slice parallel to the y-axis. Thus, the integral is going to be done dx so r and h have to be in terms of x.

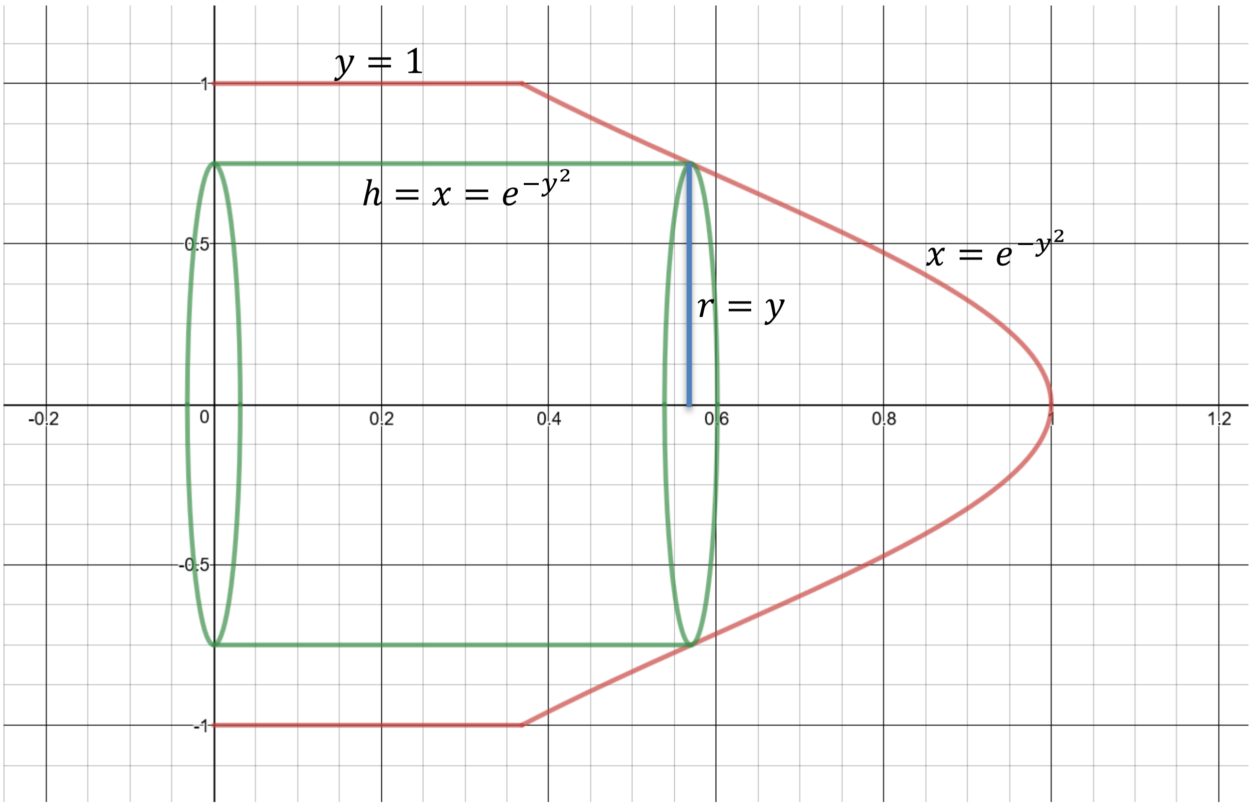
$$h = y_1 - y_2 = x - x^2$$
$$r = x$$

$$V = \int_{x=0}^{x=1} 2\pi x(x - x^2)\ dx = 2\pi \int_{x=0}^{x=1} (x^2 - x^3)\ dx$$

$$= 2\pi\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Bigg|_{x=0}^{x=1} = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= 2\pi\left(\frac{1}{12}\right) = \frac{\pi}{6}.$$

Ex. Find the volume of the region formed by revolving the region bounded by $x = e^{-y^2}$, the y-axis, $y = 0$, and $y = 1$ about the x-axis.



We're slicing parallel to the x-axis so the volume integral will be dy. Thus, we must find r and h in terms of y.

$$h = x = e^{-y^2}$$
$$r = y$$

$$V = \int_{y=0}^{y=1} 2\pi r y \; dy = \int_{y=0}^{y=1} 2\pi y e^{-y^2} \, dy$$

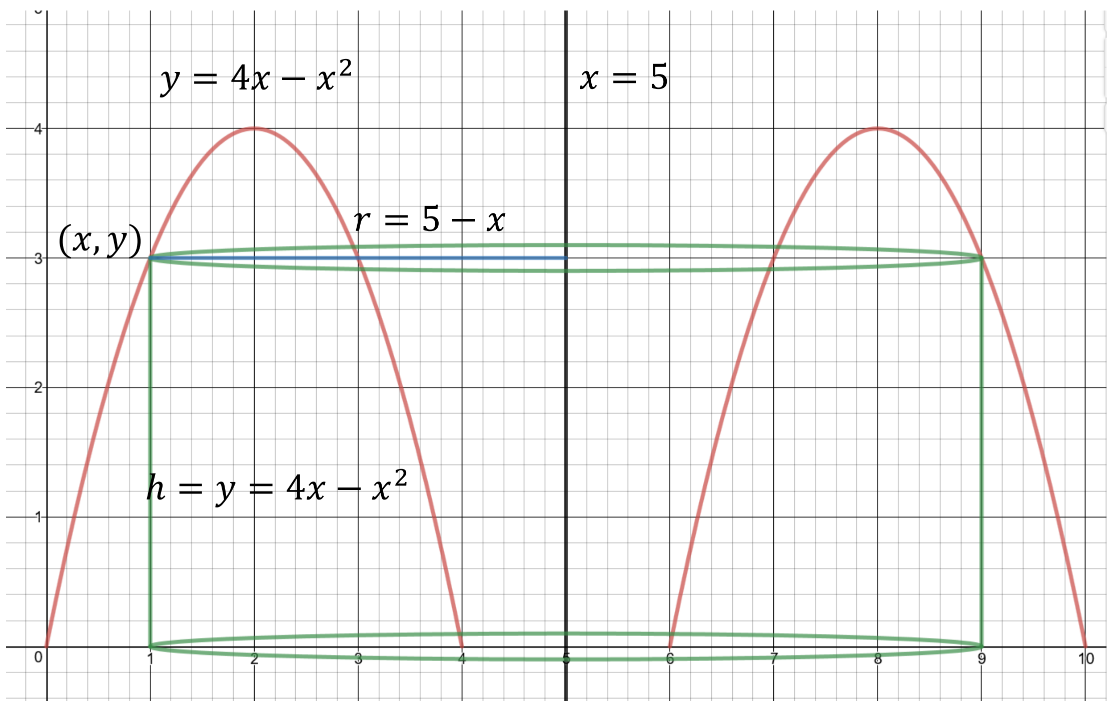
$$= -\pi e^{-y^2}\Big|_{y=0}^{y=1}$$

You get the above antiderivative by letting $u = -y^2$.

$$= -\pi(e^{-1} - e^{0})$$

$$= \pi\left(1 - \frac{1}{e}\right).$$

Ex. Using cylindrical shells, find the volume of the solid obtained by rotating about the line $x = 5$ the region bounded by $y = 4x - x^2$ and the x-axis.



$y = 4x - x^2$ intersects the x-axis when:

$$4x - x^2 = 0$$
$$x(4x - x) = 0$$
$$x = 0; \quad x = 4$$

We're slicing parallel to the y-axis, so we will be integrating dx. Thus, we need r and h in terms of x.

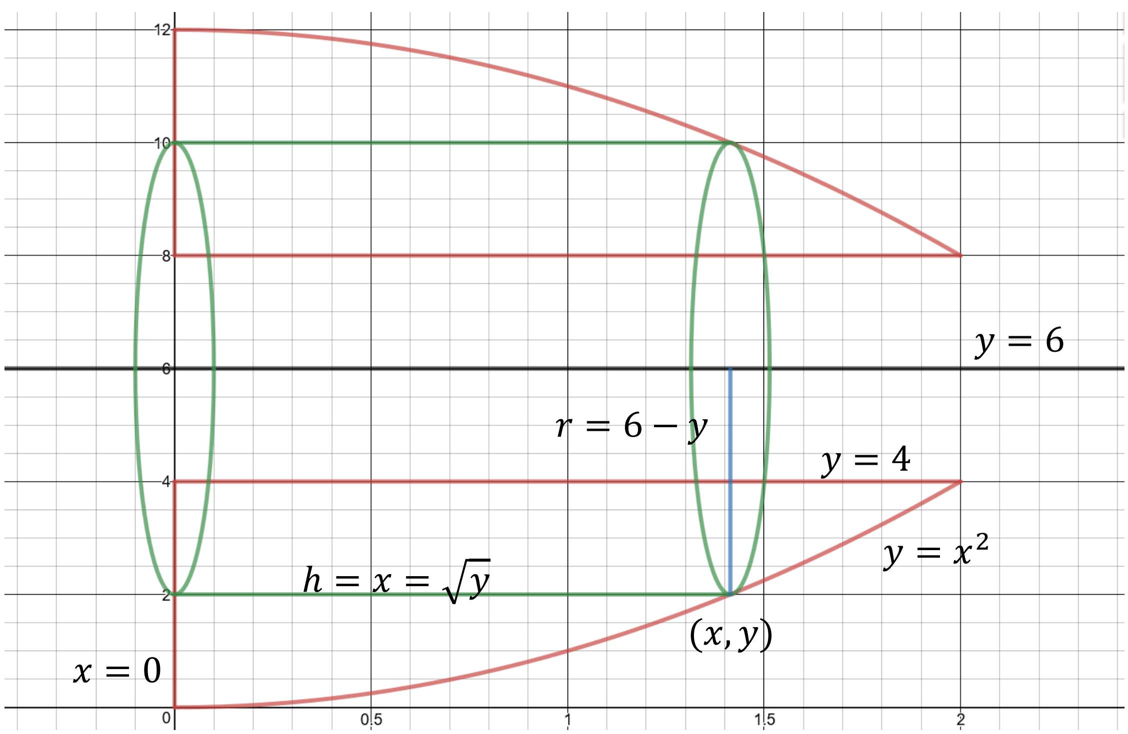
$$r = 5 - x$$
$$h = 4x - x^2$$

$$V = \int_{x=0}^{x=4} 2\pi rh \ dx = \int_{x=0}^{x=4} 2\pi(5-x)(4x-x^2)\, dx$$

$$= 2\pi \int_{x=0}^{x=4} (20x - 9x^2 + x^3)\, dx = 2\pi \left(10x^2 - 3x^3 + \frac{x^4}{4}\right)\Bigg|_{x=0}^{x=4}$$

$$= 2\pi \left(10(4)^2 - 3(4)^3 + \frac{4^4}{4}\right) = 64\pi.$$

Ex. Use cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $y = x^2$, $x = 0$, and $y = 4$ about the line $y = 6$.



We're slicing parallel to the x-axis so the integral will be done dy. Thus, we must find r and h in terms of y.

$$r = 6 - y$$
$$h = x = \sqrt{y}$$

$$V = \int_{y=0}^{y=4} 2\pi rh\ dy = \int_{y=0}^{y=4} 2\pi(6-y)(y^{\frac{1}{2}})\,dy$$

$$= 2\pi \int_{y=0}^{y=4} \left(6y^{\frac{1}{2}} - y^{\frac{3}{2}}\right) dy = 2\,\pi \left[(6)\frac{2}{3}y^{\frac{3}{2}} - \frac{2}{5}y^{\frac{5}{2}}\right]\Bigg|_{y=0}^{y=4}$$

$$= 2\pi\left(4(4)^{\frac{3}{2}} - \tfrac{2}{5}(4)^{\frac{5}{2}}\right)$$

$$= 2\pi\left(4(8) - \tfrac{2}{5}(32)\right)$$

$$= 2\pi\left(32 - \frac{64}{5}\right) = \frac{192\pi}{5}\,.$$