Approximating the Value of Integrals

We know from The Fundamental Theorem of Calculus that:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F'(x) = f(x)$.

But how do we evaluate $\int_a^b f(x)\,dx$ if there is no elementary function, $F(x)$, such that $F'(x) = f(x)$?

For example:

$$\int_0^2 e^{(x^2)}\,dx \quad \text{or} \quad \int_0^1 \sqrt{1-x^3}\,dx.$$

In cases like these, we need to approximate the values of the integral.

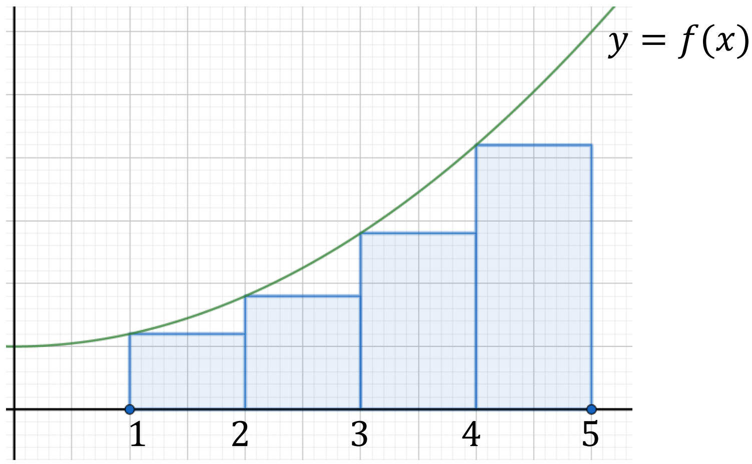
One way to approximate the value of a definite integral is with Riemann sums. That is, we break the interval $[a, b]$ into n equal subintervals each of length $\Delta x = \frac{b-a}{n}$ and choose a point x_i^* out of each interval and:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Three common choices for x_i^* are the left endpoint, the right endpoint, or the midpoint.

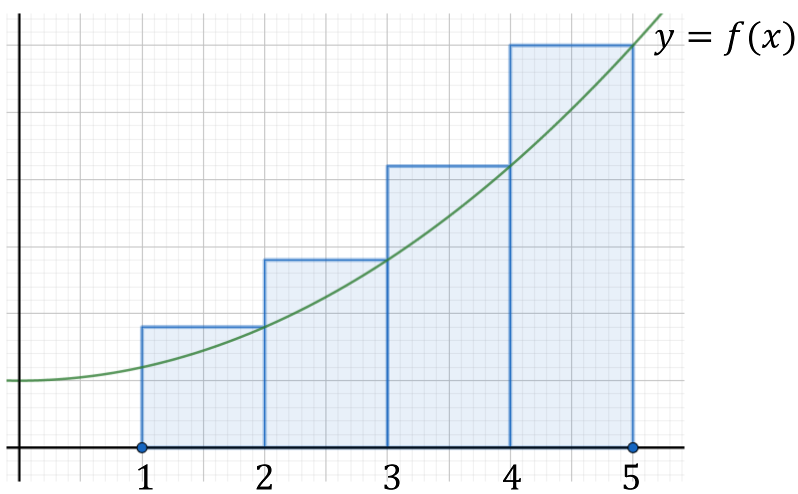
Using the left endpoint:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$



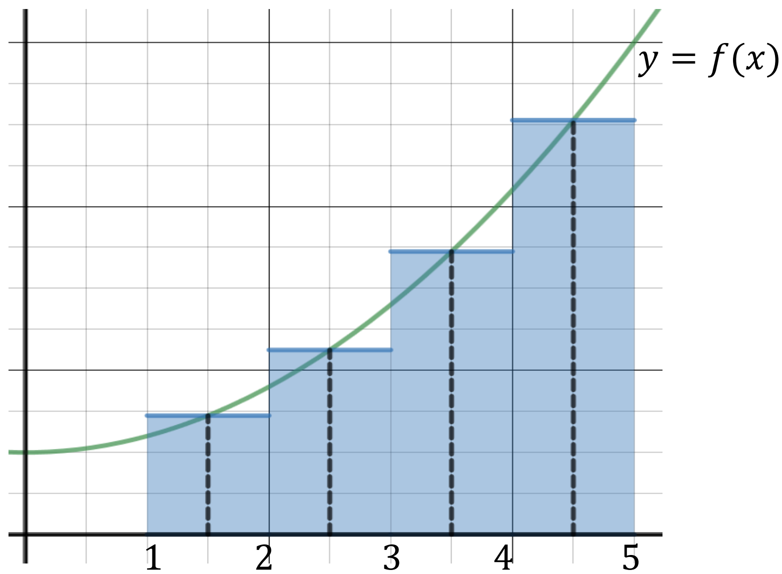
Using the right endpoint:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$



If $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$, then using this midpoint:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x$$



In general,

Left endpoint method:

$$\int_a^b f(x)\,dx \approx L_n \approx \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

Right endpoint method:

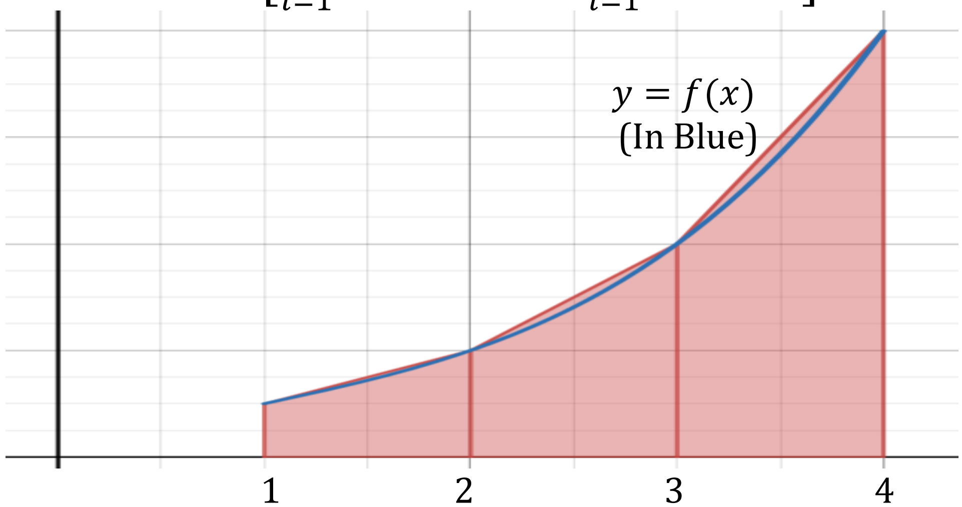
$$\int_a^b f(x)\,dx \approx R_n \approx \sum_{i=1}^{n} f(x_i)\,\Delta x$$

Midpoint method:

$$\int_a^b f(x)\,dx \approx M_n \approx \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x$$

Notice that the left endpoint, right endpoint, and midpoint methods are all using areas of rectangles to approximate $\int_a^b f(x)\,dx$. However, another approach is to use areas of trapezoids to approximate $\int_a^b f(x)\,dx$.

$$\int_a^b f(x)\,dx \approx \frac{1}{2}\left[\sum_{i=1}^{n} f(x_{i-1})\,\Delta x + \sum_{i=1}^{n} f(x_i)\,\Delta x\right]$$



The trapezoidal rule is an average of the left midpoint and right midpoint methods.

$$\int_a^b f(x)\,dx \approx \frac{1}{2}\left[\sum_{i=1}^{n} f(x_{i-1})\,\Delta x + \sum_{i=1}^{n} f(x_i)\,\Delta x\right]$$

$$= \frac{\Delta x}{2}\left[\sum_{i=1}^{n} f(x_{i-1}) + \sum_{i=1}^{n} f(x_i)\right]$$

$$= \frac{\Delta x}{2}\left[\sum_{i=1}^{n} \left(f(x_{i-1}) + f(x_i)\right)\right]$$

$$= \frac{\Delta x}{2}\left[\big(f(x_0) + f(x_1)\big) + \big(f(x_1) + f(x_2)\big) + \cdots + \big(f(x_{n-1}) + f(x_n)\big)\right]$$

$$= \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]\ .$$

Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx T_n$$

$$= \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)].$$

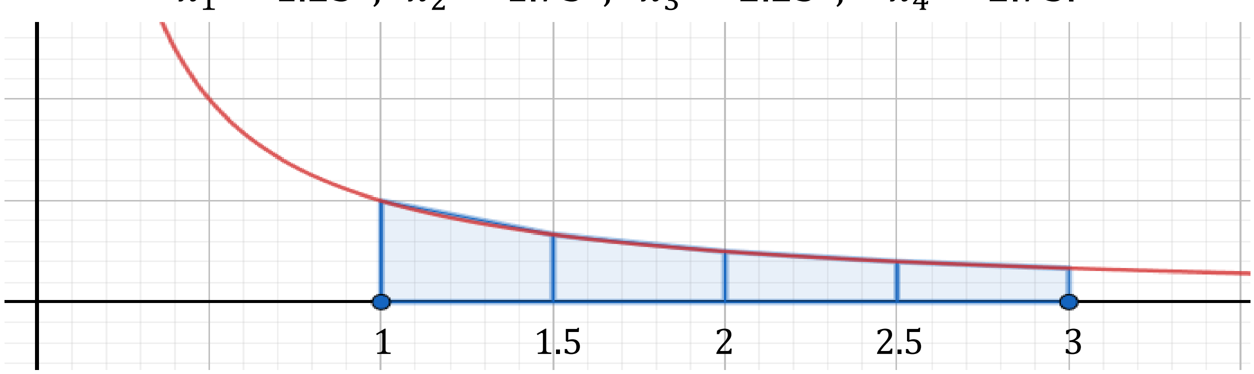
Ex. Approximate $\int_1^3 \frac{1}{x}\,dx$ using the midpoint rule, the left endpoint rule, the right endpoint rule, and the trapezoidal rule with $n = 4$.

For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$. So we have:

$x_0 = 1, \quad x_1 = 1.5, \quad x_2 = 2, \quad x_3 = 2.5, \quad x_4 = 3.$

So the midpoints occur at:

$\bar{x}_1 = 1.25\ ,\ \bar{x}_2 = 1.75\ ,\ \bar{x}_3 = 2.25\ ,\ \bar{x}_4 = 2.75.$



$$M_4 = \sum_{i=1}^{4} f(\bar{x}_i)\,\Delta x = [f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + f(\bar{x}_4)](0.5)$$

$$= [f(1.25) + f(1.75) + f(2.25) + f(2.75)](0.5)$$

$$= \left[\frac{1}{1.25} + \frac{1}{1.75} + \frac{1}{2.25} + \frac{1}{2.75}\right](0.5)$$

$$\approx 1.11248$$

$$L_4 = \sum_{i=1}^{4} f(x_{i-1})\,\Delta x = [f(x_0) + f(x_1) + f(x_2) + f(x_3)](0.5)$$

$$= [f(1.0) + f(1.5) + f(2.0) + f(2.5)](0.5)$$

$$= \left[\frac{1}{1.0} + \frac{1}{1.5} + \frac{1}{2.0} + \frac{1}{2.5}\right](0.5)$$

$$\approx 1.28333$$

$$R_4 = \sum_{i=1}^{4} f(x_i)\,\Delta x = [f(x_1) + f(x_2) + f(x_3) + f(x_4)](0.5)$$

$$= [f(1.5) + f(2.0) + f(2.5) + f(3.0)](0.5)$$

$$= \left[\frac{1}{1.5} + \frac{1}{2.0} + \frac{1}{2.5} + \frac{1}{3.0}\right](0.5)$$

$$= 0.95000$$

$$T_4 = \frac{\Delta x}{2}[f(x_0 + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$$

$$= \frac{0.5}{2}[f(1) + 2f(1.5) + 2f(2.0) + 2f(2.5) + f(3)]$$

$$= \frac{0.5}{2}\left[\frac{1}{1} + 2\left(\frac{1}{1.5}\right) + 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{2.5}\right) + \frac{1}{3}\right]$$

$$\approx 1.11667$$

In this case, we know how to calculate:

$$\int_1^3 \frac{1}{x}\,dx = \ln x\Big|_1^3 = \ln 3 - \ln 1 = \ln 3 \approx 1.09861$$

Error using the midpoint rule:

$$E_M \approx 1.09861 - 1.11248 = -0.01387$$

Error using the left endpoint rule:

$$E_L \approx 1.09861 - 1.28333 = -0.18472$$

Error using the right endpoint rule:

$$E_R \approx 1.09861 - 0.95000 = 0.14861$$

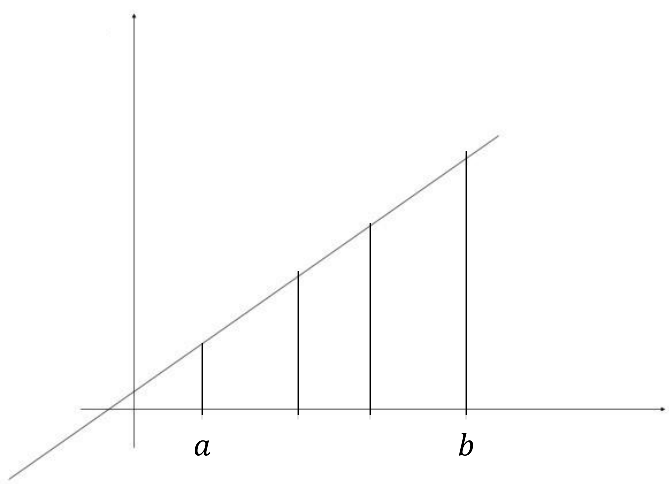
Error using the trapezoidal rule:

$$E_T \approx 1.09861 - 1.11667 = -0.01806$$

The more subdivisions we take of the intervals, the better the approximation will tend to be with all of the methods. In general, the trapezoidal and midpoint rules tend to be more accurate than either the left or right endpoint rules. Also the midpoint rule tends to be more accurate than the trapezoidal rule.

Notice that if $f(x)$ is a linear function, then both the trapezoidal rule and the midpoint rule will be exactly equal to the integral (not just an approximation).

$$\int_a^b f(x) = T_N = M_N$$



Thus when estimating the error in each approximation

$$E_T = \int_a^b f(x)\,dx - T_n$$

$$E_M = \int_a^b f(x)\,dx - M_n$$

E_T and E_M depend on $f''(x)$ (if $f(x)$ is a linear function $f''(x) = 0$).

Error bounds: Suppose $|f''(x)| \leq k$ for all $a \leq x \leq b$. If E_T and E_M are the errors in the trapezoidal and midpoint rules, then:

$$|E_T| \leq \frac{k(b-a)^3}{12n^2} \qquad |E_M| \leq \frac{k(b-a)^3}{24n^2}$$

Ex. Use the error bounds to approximate the error in $\int_1^3 \left(\frac{1}{x}\right) dx$ using the trapezoidal method and the midpoint method with $n = 4$.

$$f(x) = \frac{1}{x}$$
$$f'(x) = -\frac{1}{x^2}$$
$$f''(x) = \frac{2}{x^3}$$

When $1 \leq x \leq 3$:

$$\left|\frac{2}{x^3}\right| \leq \frac{2}{1^3} = 2 = k.$$

We now know $a = 1, b = 3, n = 4$.

$$|E_T| \leq \frac{k(b-a)^3}{12n^2} \leq \frac{2(3-1)^3}{12(4)^2} = \frac{2(8)}{12(16)} = \frac{1}{12} \approx 0.08333$$

$$|E_M| \leq \frac{k(b-a)^3}{24n^2} \leq \frac{2(3-1)^3}{24(4)^2} = \frac{1}{24} \approx 0.04167.$$

We saw earlier that in this case:

$$|E_T| = 0.01806$$

$$|E_M| = 0.01387$$

Ex. How large should we take n in order to guarantee that the error in the previous example is less than 0.001 for the following:

a) The trapezoidal rule

b) The midpoint rule

a) $|E_T| \leq \frac{k(b-a)^3}{12n^2} = \frac{2(3-1)^3}{12n^2} = \frac{16}{12n^2} < 0.001$

$$\frac{1}{n^2} < (0.001)\left(\frac{12}{16}\right) = 0.00075$$

$$n^2 > \frac{1}{0.00075} \approx 1{,}333.33$$

$$n > 36.5$$

So take $n \geq 37$.

b) $|E_M| \leq \frac{k(b-a)^3}{24n^2} = \frac{2(3-1)^3}{12n^2} = \frac{16}{24n^2} < 0.001$

$$\frac{1}{n^2} < (0.001)\left(\frac{24}{16}\right) = 0.0015$$

$$n^2 > \frac{1}{0.0015} \approx 666.67$$

$$n > 25.8$$

So take $n \geq 26$.

Ex. Approximate $\int_0^1 e^{x^2}\, dx$.

a) Use the midpoint rule and the trapezoidal rule each with $n = 5$
b) Give an upper bound for the error in these approximations

a) By the midpoint rule:

$$\int_0^1 e^{x^2}\, dx \approx \Delta x \sum_{i=1}^{5} f(\bar{x}_i)$$

$$a = 0\,,\ \ b = 1,\ \ n = 5 \ \ \Rightarrow \Delta x = \frac{b-a}{n} = \frac{1-0}{5} = .2$$

$$x_0 = 0, \quad x_1 = 0.2, \quad x_2 = 0.4, \quad x_3 = 0.6, \quad x_4 = 0.8, \quad x_5 = 1.0$$

$$\bar{x}_1 = 0.1, \quad \bar{x}_2 = 0.3, \quad \bar{x}_3 = 0.5, \quad \bar{x}_4 = 0.7, \quad \bar{x}_5 = 0.9$$

$$\int_0^1 e^{x^2}\, dx \approx .2[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)]$$

$$= .2[e^{0.01} + e^{0.09} + e^{0.25} + e^{0.49} + e^{0.81}] \approx 1.45369.$$

By the trapezoidal rule:

$$\int_0^1 e^{x^2}\,dx \approx \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)]$$

$$= \frac{\Delta x}{2}[f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1)]$$

$$= \frac{0.2}{2}[e^0 + 2e^{0.04} + 2e^{0.16} + 2e^{0.36} + 2e^{0.64} + e^1]$$

$$\approx 1.48065.$$

b) $f(x) = e^{x^2}$

$f'(x) = 2xe^{x^2}$

$f''(x) = 2x\left(2xe^{x^2}\right) + 2xe^{x^2} = (4x^2 + 2)e^{x^2}$

$|E_M| \le \frac{k(b-a)^3}{24n^2}$ where $|f''(x)| \le k$ for $a \le x \le b$

$$a = 0,\ \ b = 1,\ \ n = 5$$

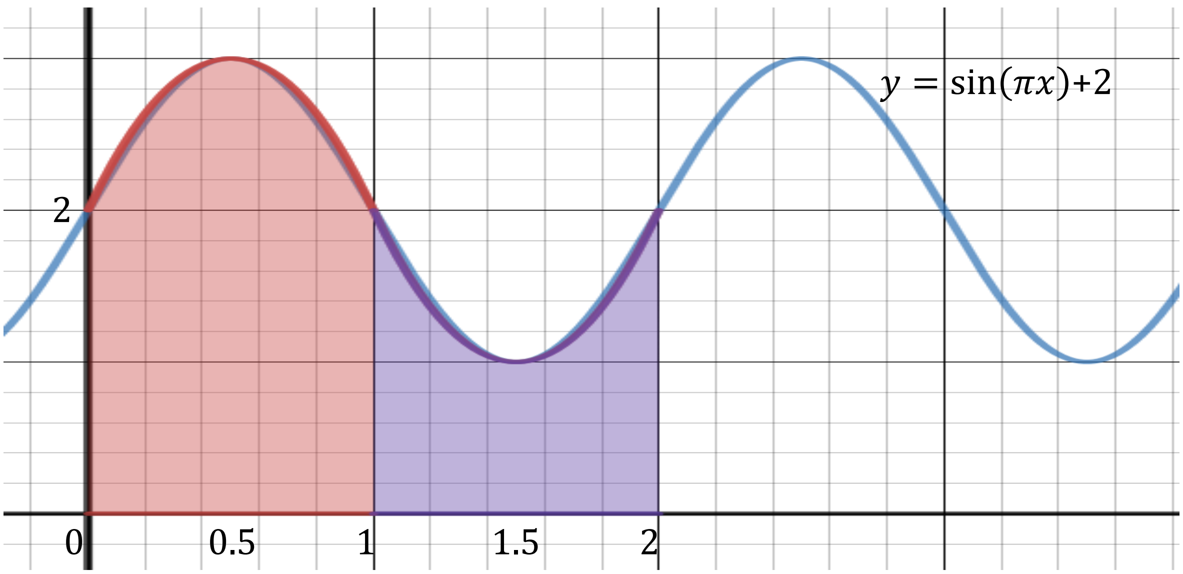
$|f''(x)| = \left|(4x^2 + 2)e^{x^2}\right| \le 6e$ (when $x = 1$)

$$|E_M| \le \frac{k(b-a)^3}{24n^2} \le \frac{(6e)(1)^3}{24(5)^2} \approx 0.02718$$

$$|E_T| \le \frac{k(b-a)^3}{12n^2} = \frac{6e(1)^3}{12(5)^2} \approx 0.05437.$$

Simpson's Rule

Simpson's Rule uses parabolas instead of line segments to approximate the curve $y = f(x)$. This time we divide $[a, b]$ into an even number of subintervals. On each consecutive pair of intervals approximate the curve with a parabola.



It can be shown that the area under the parabola through the points $(x_0, y_0), (x_1, y_1)$, and (x_2, y_2) is given by $\frac{\Delta x}{3}(y_0 + 4y_1 + y_2)$ where $\Delta x = \frac{b-a}{n}$ and n is even.

This leads us to Simpson's Rule:

$$\int_a^b f(x)\,dx \approx S_n$$

$$= \frac{\Delta x}{3}\begin{bmatrix} f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots \\ \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \end{bmatrix}$$

Ex. Use Simpson's Rule with $n = 4$ to approximate $\int_1^3 \left(\frac{1}{x}\right) dx$.

$$a = 1, \quad b = 3, \quad \Delta x = \frac{b-a}{n} = \frac{3-1}{4} = 0.5$$

$$x_0 = 1.0, \quad x_1 = 1.5, \quad x_2 = 2, \quad x_3 = 2.5, \quad x_4 = 3.0$$

$$\int_1^3 \frac{1}{x} dx \approx S_4 = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4(x_3) + f(x_4)]$$

$$= \frac{0.5}{3}[f(1.0) + 4f(1.5) + 2f(2.0) + 4f(2.5) + f(3.0)]$$

$$= \frac{0.5}{3}\left[\frac{1}{1} + 4\left(\frac{1}{1.5}\right) + 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{2.5}\right) + \frac{1}{3}\right]$$

$$= 1.10000.$$

Notice that $\int_1^3 \frac{1}{x} dx = \ln 3 \approx 1.09861.$

Our approximation using M_4 was 1.11248 and using T_4 was 1.11667.

The approximation using $S_4 = 1.10000$ is clearly much better.

Error bound for Simpson's Rule:

Suppose $\left|f^{(4)}(x)\right| \leq k$ for $a \leq x \leq b$. If E_S is the error in Simpson's Rule, then:

$$|E_S| \leq \frac{k(b-a)^5}{180n^4}.$$

Ex. How large should we take n to guarantee that Simpson's Rule approximation of $\int_1^3 \frac{1}{x}\,dx$ is accurate to within 0.001?

$$\begin{aligned} f(x) &= x^{-1} \\ f'(x) &= -x^{-2} \\ f''(x) &= 2x^{-3} \\ f'''(x) &= -6x^{-4} \\ f''''(x) &= 24x^{-5} \end{aligned}$$

$$\left|f^{(4)}(x)\right| = \left|\frac{24}{x^5}\right| \le \frac{24}{1} = 24 = k \quad \text{when } 1 \le x \le 3$$

$$|E_S| = \frac{k(b-a)^5}{180n^4} = \frac{24(3-1)^5}{180n^4} < 0.001 \quad \Rightarrow \quad \frac{768}{180n^4} < 0.001$$

$$\frac{1}{n^4} < \frac{180}{768}(0.001) \approx 0.0002349$$

$$n^4 > \frac{1}{0.0002349}$$

$$n > \left(\frac{1}{0.0002349}\right)^{\frac{1}{4}} \approx 8.1$$

So take $n = 10$ (n must be even).

Notice that when using the trapezoidal rule we needed $n = 37$ and for the midpoint rule we needed $n = 26$ to guarantee an error in $\int_1^3 \frac{1}{x}\,dx$ to be less than 0.001.

Ex. Use Simpson's Rule with $n = 4$ to approximate $\int_0^1 e^{x^2}\,dx$. Estimate the error involved in this approximation.

$$a = 0, \quad b = 1, \quad \Delta x = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

$$x_0 = 0, \quad x_1 = 0.25, \quad x_2 = 0.50, \quad x_3 = 0.75, \quad x_4 = 1$$

$$\int_0^1 e^{x^2}\,dx \approx S_4 = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)]$$

$$= \frac{0.25}{3}[f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + f(1)]$$

$$= \frac{0.25}{3}\left[e^0 + 4\left(e^{(0.25)^2}\right) + 2\left(e^{(0.5)^2}\right) + 4\left(e^{(0.75)^2}\right) + e^1\right]$$

$$\approx 1.46371$$

$$|E_S| \le \frac{k(b-a)^5}{180n^4}; \qquad f(x) = e^{x^2}$$

By taking successive derivatives we get: $f^{(4)}(x) = (12 + 48x^2 + 16x^4)e^{x^2}$.

Since $0 \le x \le 1$, $f^{(4)} \ge 0$ and it takes its maximum when $x = 1$ since it's an increasing function.

$$\left|f^{(4)}(x)\right| \le \left(12 + 48 + 16)e^{(1)^2}\right) = 76e$$

$$|E_S| \le \frac{76e(1-0)^5}{180(4)^4} \approx 0.00448.$$

Notice that the error is much less than the error bound for the trapezoidal rule, $n = 5,\ 0.05437$, and the error bound for the midpoint rule, $n = 5,\ 0.02718$.