Inverse Functions

Recall that the **domain** of a function, f, is the set of values (which will be real numbers in this case) that we can "plug into" the function. The **range** of a function, f, is the set of all values that the function f takes on.

Ex. Identify the domain and range for f and g below.

x	$f(x)$
1	-2
3	$\frac{1}{2}$
5	0
7	3
9	1

x	$g(x)$
$\frac{1}{6}$	-3
$\frac{1}{5}$	2
$\frac{1}{4}$	-3
$\frac{1}{3}$	$\frac{2}{3}$

Domain of $f = \{1, 3, 5, 7, 9\}$

Range of $f = \left\{-2, 0, \frac{1}{2}, 1, 3\right\}$

Domain of $g = \left\{\frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}\right\}$

Range of $g = \left\{-3, \frac{2}{3}, 2\right\}$.

Ex. Let $F(x) = \sqrt{x-1}$ and $G(x) = x^2$. Find the natural domain (i.e. the largest subset of the real numbers, $\mathbb{R}$) and range of F and G.

In order for $\sqrt{x-1}$ to be a real number we must require $x - 1 \geq 0$.

Thus the domain of $F = \{x \in \mathbb{R} | x \geq 1\}$.

$F(1) = \sqrt{1-1} = 0$ and $F(x)$ is a continuous, increasing function where: $\lim_{x\to\infty} F(x) = \infty$.

So we have range of $F = \{y \in \mathbb{R} | \ y \geq 0\}$.

The domain of $G = \{x \in \mathbb{R}\}$ since there are no restrictions on which real numbers can be "plugged into" G.

The range of $G = \{y \in \mathbb{R} |\ y \geq 0\}$ since $x^2 \geq 0$ and:

$$\lim_{x \to \pm\infty} x^2 = \infty$$

Def. A function, f, is called a **one-to-one function** if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for any x_1 and x_2 in the domain of f.

Notice that a function is one-to-one when you can't have two different elements of the domain have the same value $f(x)$.

Ex. Determine if f or g is a one-to-one function.

x	$f(x)$
1	-2
3	$\frac{1}{2}$
5	0
7	3
9	1

x	$g(x)$
$\frac{1}{6}$	-3
$\frac{1}{5}$	2
$\frac{1}{4}$	-3
$\frac{1}{3}$	$\frac{2}{3}$

f is a one-to-one function since no two domain values has the same $f(x)$.

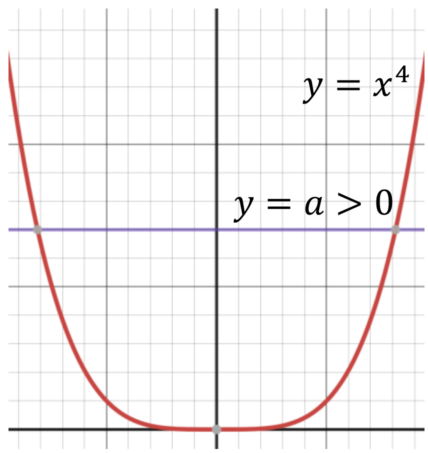
g is not a one-to-one function since $g\left(\frac{1}{6}\right) = g\left(\frac{1}{4}\right) = -3$.

Horizontal Line Test: A function is one-to-one if, and only if, no horizontal line intersects its graph more than once.

Ex. Show $F(x) = x^4$ is not one-to-one.

Solution 1: $F(x)$ is not one-to-one because $F(-1) = F(1) = 1$

Solution 2:

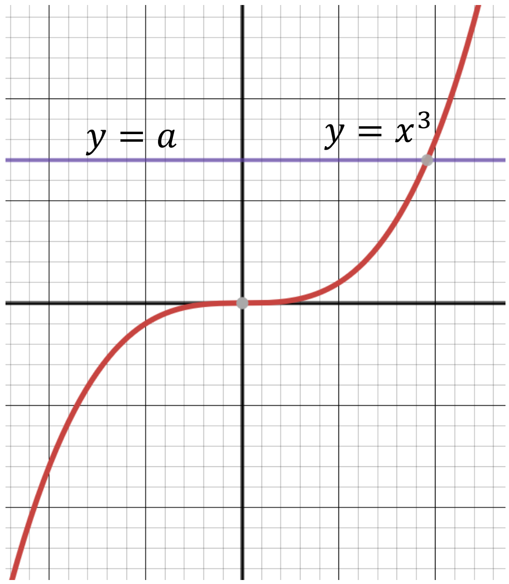


Notice that any horizontal line $y = a$, where $a > 0$, will intersect the graph in more than one point. Thus, $F(x) = x^4$ is not one-to-one.

Ex. Show $h(x) = x^3$ is one-to-one.

Solution 1: We need to show that if $h(x_1) = h(x_2)$, then $x_1 = x_2$.
We know: $x_1^3 = x_2^3 \Rightarrow x_1 = x_2$ so $h(x)$ is one-to-one.

Solution 2:



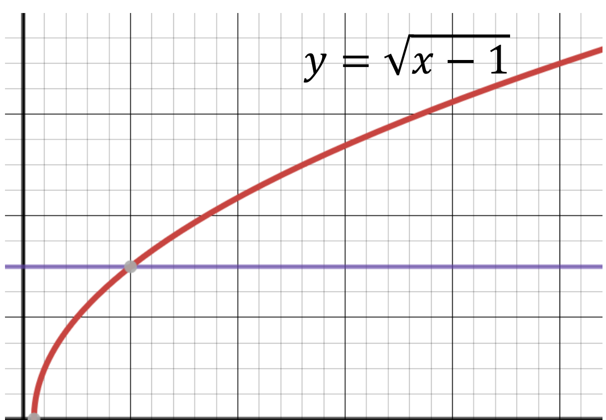
We can see from the graph of $h(x) = x^3$ that every horizontal line intersects the graph in exactly one point. This means that no horizontal line will intersect the graph in more than one point.

Ex. Show that $F(x) = \sqrt{x-1}$ is one-to-one.

Solution 1: $F(x_1) = F(x_2)$ means $\sqrt{x_1 - 1} = \sqrt{x_2 - 1}$

$\Longrightarrow \quad x_1 - 1 = x_2 - 1 \quad \Rightarrow \quad x_1 = x_2$ so F is one-to-one.

Solution 2:



Notice that no horizontal line intersects the graph in more than one point. Thus, $F(x)$ is one-to-one.

Def. Let f be a one-to-one function with domain A and range B.
We can then say that its **inverse function** f^{-1} has domain B, range A, and is defined by: $f^{-1}(y) = x$ if, and only if, $y = f(x)$ for any $y \in B$.

Another way of looking at inverse functions is that an inverse function f^{-1} "undoes" what the original function does. Meaning if we start with any point $x \in A =$ domain of f, then we have:

$$x \xrightarrow{f} f(x) \xrightarrow{f^{-1}} f^{-1}\big(f(x)\big) = x$$

So f^{-1} brings us right back to the original value x.

Ex. Given the function f below, find the inverse function f^{-1} and describe the domain and range of f^{-1}.

x	$f(x)$		$f^{-1}(x)$
1	-2		$f^{-1}(-2) = 1$
3	$\frac{1}{2}$		$f^{-1}\left(\frac{1}{2}\right) = 3$
5	0		$f^{-1}(0) = 5$
7	3		$f^{-1}(3) = 7$
9	1		$f^{-1}(1) = 9$

Domain of $f^{-1}(x) = \left\{-2, 0, \frac{1}{2}, 1, 3\right\} =$ range of f

Range of $f^{-1}(x) = \{1, 3, 5, 7, 9\} =$ domain of f.

In fact, we will always have:

$$\text{Domain of } f^{-1} = \text{Range of } f$$
$$\text{Range of } f^{-1} = \text{Domain of } f$$
$$f\left(f^{-1}(x)\right) = x$$
$$f^{-1}\left(f(x)\right) = x.$$

Remember that we can only have inverse functions for functions that are one-to-one. But if we do have a one-to-one function, then how can we find its inverse? In general, it isn't always possible to write down an explicit formula for an inverse function. However, if we have $y = f(x)$ and we can solve this equation for x in terms of y, then we get an explicit formula for the inverse function by switching the roles of x and y.

Ex. Find the inverse function for $f(x) = 3x - 2$, start by solving for x.

$$y = 3x - 2$$
$$y + 2 = 3x$$
$$\frac{1}{3}y + \frac{2}{3} = x$$

Now switch the roles of x and y:

$$y = \frac{1}{3}x + \frac{2}{3}$$
$$f^{-1}(x) = \frac{1}{3}x + \frac{2}{3}.$$

Ex. Find a formula for the inverse function of $y = \frac{3x+1}{1-2x}$.

First, solve for x in terms of y.

$$y = \frac{3x+1}{1-2x}$$
$$y(1-2x) = 3x + 1$$
$$y - 2xy = 3x + 1$$
$$y - 1 = 3x + 2xy$$
$$y - 1 = x(3 + 2y)$$
$$\frac{y-1}{3+2y} = x$$

Now switch x and y:

$$f^{-1}(x) = \frac{x-1}{3+2x}$$

Ex. Find the inverse function of $f(x) = x^3 - 8$.

$$y = x^3 - 8$$
$$y + 8 = x^3$$
$$\sqrt[3]{y+8} = x$$

Now switch the roles of x and y:

$$y = \sqrt[3]{x+8}$$
$$f^{-1}(x) = \sqrt[3]{x+8}\,.$$

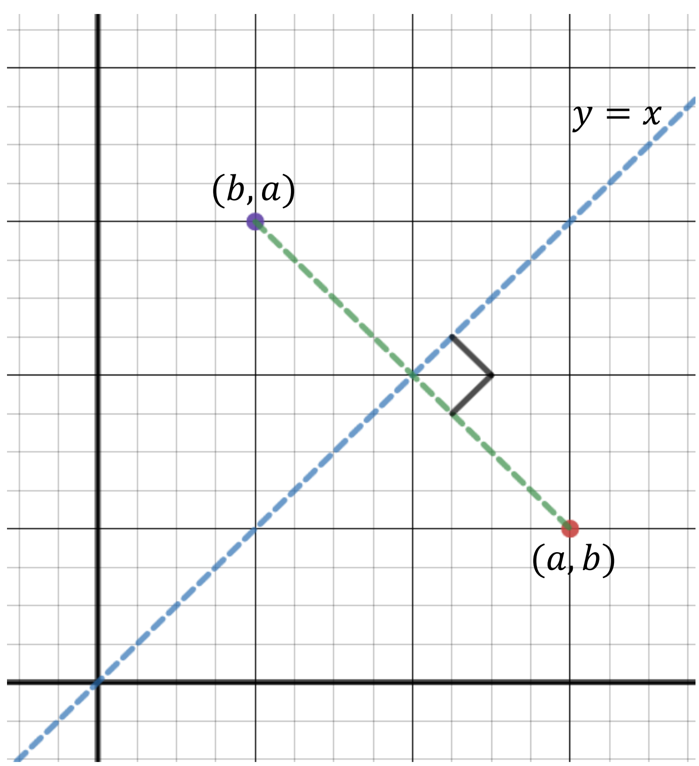
Ex. Using the previous example, show that $f\left(f^{-1}(x)\right) = x$ and $f^{-1}\left(f(x)\right) = x$.

$$f(x) = x^3 - 8 \qquad\qquad f^{-1}(x) = \sqrt[3]{x+8}$$

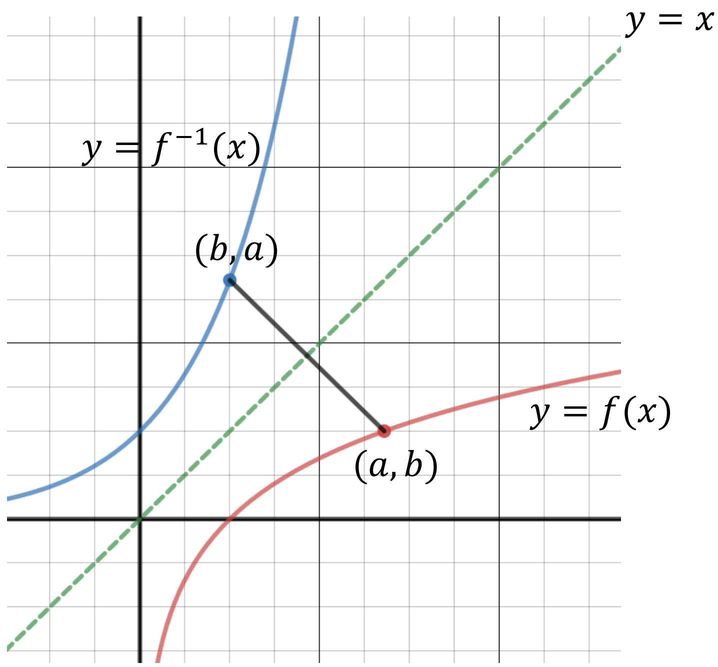
$$f\left(f^{-1}(x)\right) = f\left(\sqrt[3]{x+8}\right) = \left(\sqrt[3]{x+8}\right)^3 - 8 = x + 8 - 8 = x.$$

$$f^{-1}\left(f(x)\right) = f^{-1}(x^3 - 8) = \sqrt[3]{x^3 - 8 + 8} = \sqrt[3]{x^3} = x.$$

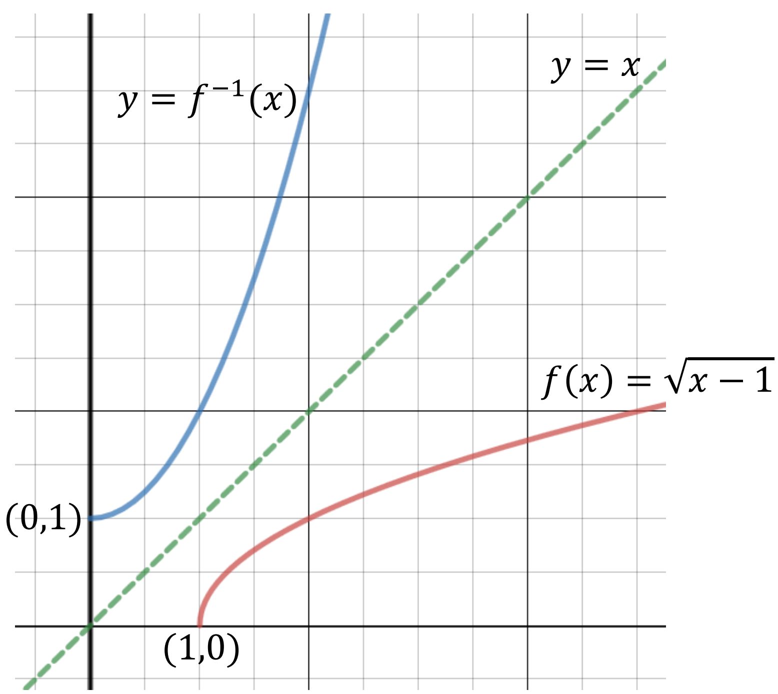
If we let $b = f(a)$, then the point (a, b) is a point on the graph of $y = f(x)$. Also, we can then say that $f^{-1}(b) = a$ so the point (b, a) is on the graph of $y = f^{-1}(x)$.



Thus, we have that the graph of f^{-1} is the reflection of the graph of $y = f(x)$ about the line $y = x$.



Ex. Given the graph of $f(x) = \sqrt{x-1}$, sketch a graph of $y = f^{-1}(x)$ below.



Theorem: If f is a one-to-one, continuous function on an interval, then its inverse function, f^{-1}, is also continuous.

Theorem: If f is a one-to-one differentiable function, with inverse function f^{-1}, and $f'\big(f^{-1}(b)\big) \neq 0$, then the inverse function is differentiable at b and:

$$(f^{-1})'(b) = \frac{1}{f'\big(f^{-1}(b)\big)}.$$

Ex. Let $f(x) = x^3 + 7x + 3\cos x$. Find $(f^{-1})'(3)$.

$$(f^{-1})'(3) = \frac{1}{f'\big(f^{-1}(3)\big)}$$

Notice that $f(0) = 3\,(\cos 0) = 3$, so $f^{-1}(3) = 0$.

$$f'(x) = 3x^2 + 7 - 3\sin x$$

$$f'\big(f^{-1}(3)\big) = f'(0) = 7$$

$$(f^{-1})'(3) = \frac{1}{f'\big(f^{-1}(3)\big)} = \frac{1}{7}.$$

Proof of Theorem: Let's let $f(a) = b,\ y = f^{-1}(x)$, and $x = f(y)$ then

$$(f^{-1})'(b) = \lim_{x \to b} \frac{f^{-1}(x) - f^{-1}(b)}{x - b}$$

$$= \lim_{y \to a} \frac{y - a}{f(y) - f(a)} = \lim_{y \to a} \frac{1}{\frac{f(y) - f(a)}{y - a}}$$

$$= \frac{1}{\lim_{y \to a} \frac{f(y) - f(a)}{y - a}} = \frac{1}{f'(a)}$$

$$= \frac{1}{f'\left(f^{-1}(b)\right)}.$$

Ex. Let $f(x) = \sqrt{x + 4}$ and $b = 3$.

a) Show that f is a one-to-one function.
b) Calculate $f^{-1}(x)$ and state the domain and range of f^{-1}.
c) Find $(f^{-1})'(3)$ by the theorem and by using part b.

a) $f(x_1) = f(x_2)$

$$\sqrt{x_1 + 4} = \sqrt{x_2 + 4}$$
$$x_1 + 4 = x_2 + 4$$
$$x_1 = x_2.$$

So f is a one-to-one function.

b) $y = \sqrt{x+4}$, solve for x in terms of y:

$$y^2 = x + 4$$
$$y^2 - 4 = x$$

Now switch x and y: $f^{-1}(x) = x^2 - 4$.

Domain of $f = \{x \in \mathbb{R} |\ x + 4 \geq 0\} = \{x \in \mathbb{R} |\ x \geq -4\}$
Range of $f = \{y \in \mathbb{R} |\ y \geq 0\}$

Domain of f^{-1} = Range of $f = \{x \in \mathbb{R} |\ x \geq 0\}$
Range of f^{-1} = Domain of $f = \{y \in \mathbb{R} |\ y \geq -4\}$

c) $(f^{-1})'(3) = \dfrac{1}{f'\left(f^{-1}(3)\right)}$

$$f(x) = 3 \quad \Rightarrow \quad \sqrt{x+4} = 3 \quad \Rightarrow \quad x + 4 = 9 \quad \Rightarrow \quad x = 5$$
$$f(5) = 3 \quad \Rightarrow \quad f^{-1}(3) = 5$$

$$f(x) = \sqrt{x+4} = (x+4)^{\frac{1}{2}}$$
$$f'(x) = \frac{1}{2}\ (x+4)^{-\frac{1}{2}}$$

$$f'(5) = \frac{1}{2}(5+4)^{-\frac{1}{2}} = \frac{1}{2}\left(\frac{1}{\sqrt{9}}\right) = \frac{1}{6}$$

$$(f^{-1})'(3) = \frac{1}{f'\left(f^{-1}(3)\right)} = \frac{1}{f'(5)} = \frac{1}{\frac{1}{6}} = 6.$$

Now using part b:

$$f^{-1}(x) = x^2 - 4$$
$$(f^{-1})'(x) = 2x$$
$$(f^{-1})'(3) = 6.$$