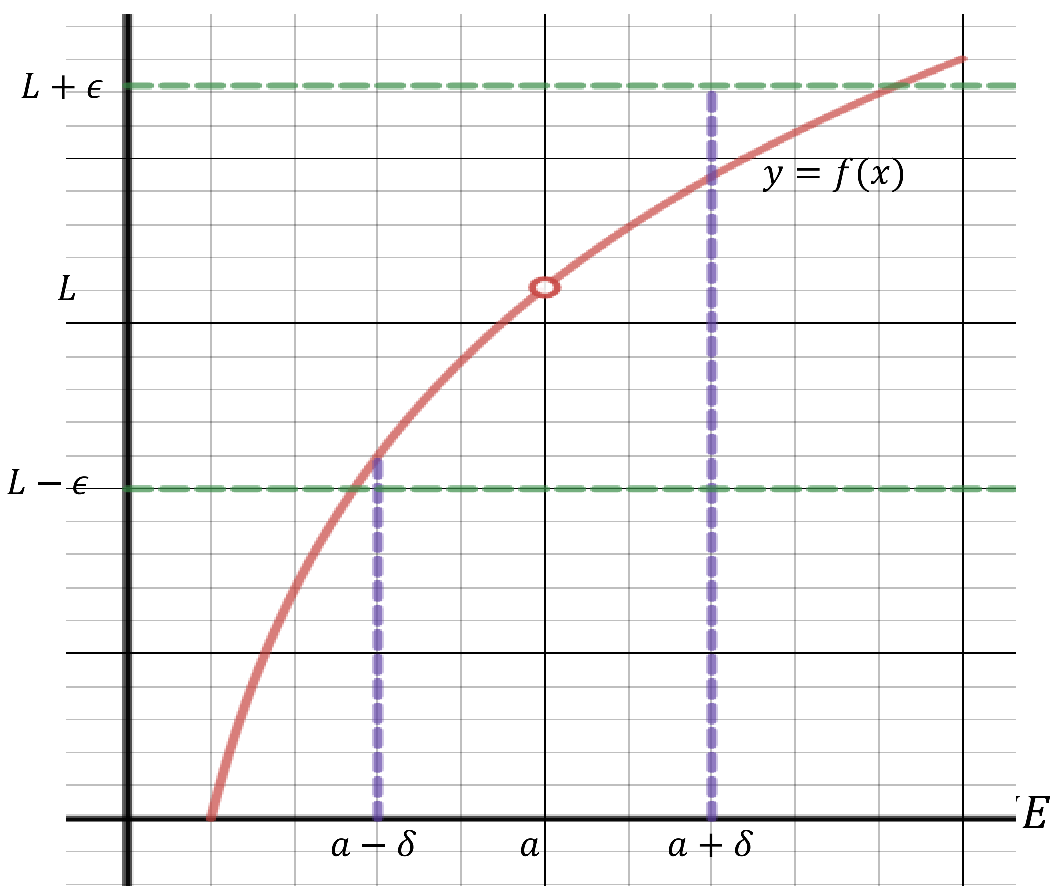
The Formal Definition of a Limit

Recall that $\lim_{x \to a} f(x) = L$ means that $f(x)$ can be forced to be arbitrarily close to L for all x sufficiently close to a (but not including $x = a$).

This means that given <u>any</u> interval around the number L, let's say $(L - \epsilon, L + \epsilon)$, we can always find an interval around the point $x = a$, let's say $(a - \delta, a + \delta)$, so that for any x (other than $x = a$), where

$a - \delta < x < a + \delta,\ f(x)$ will satisfy $L - \epsilon < f(x) < L + \epsilon$.

In general, the number δ will depend on the number ϵ.



So in order to prove that $\lim_{x \to a} f(x) = L$, we will need to show that given ANY $\epsilon > 0$ we can find a $\delta > 0$ (where δ is a function of ϵ) so that if

$a - \delta < x < a + \delta$, with $x \neq a$, then $L - \epsilon < f(x) < L + \epsilon$.

Notice that $a - \delta < x < a + \delta$, with $x \neq a$ is the same as:

$$0 < |x - a| < \delta, \text{ and}$$

$L - \epsilon < f(x) < L + \epsilon$ is the same as:

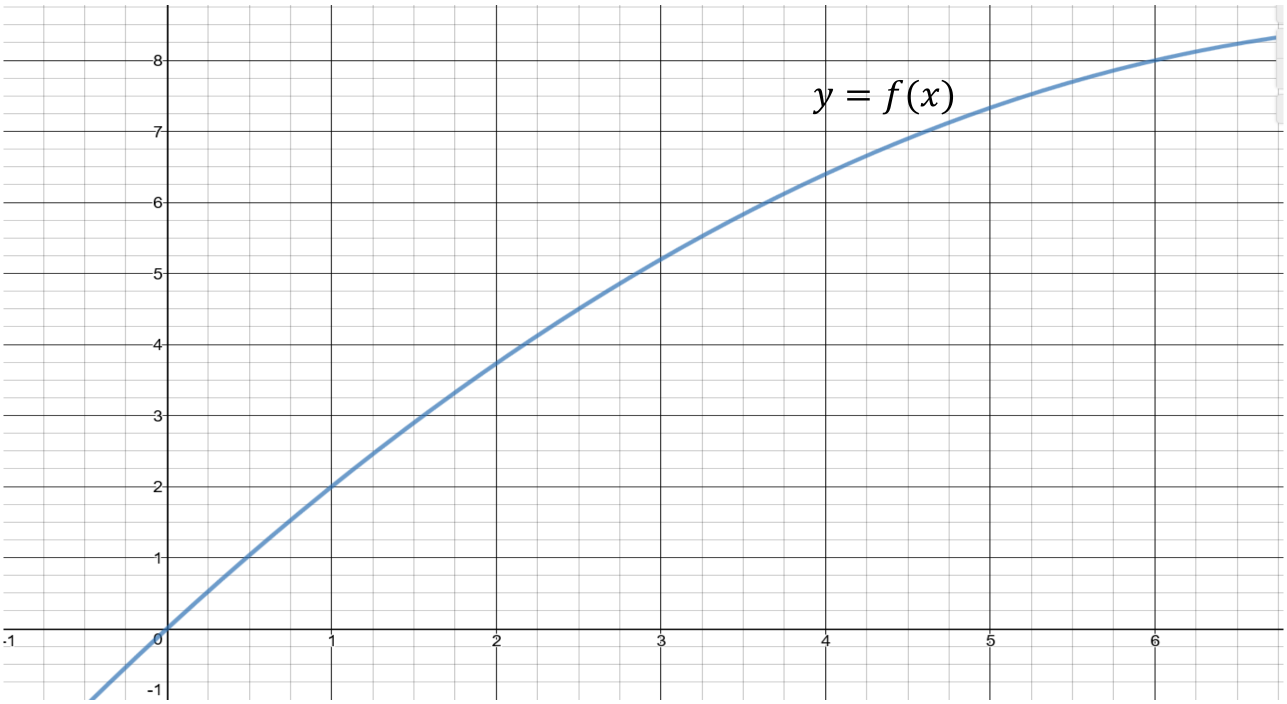
$$|f(x) - L| < \epsilon.$$

Thus one often sees the definition of $\lim_{x \to a} f(x) = L$ as

$\boldsymbol{\lim_{x \to a} f(x) = L}$ means given any $\epsilon > 0$ there exists (or we can find) a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

Ex. Suppose $|f(x) - 5| < 3$ for all x where $1 < x < 6$. Find all $\delta > 0$ such that $|f(x) - 5| < 3$ when $0 < |x - 3| < \delta$.

Start by drawing a picture.



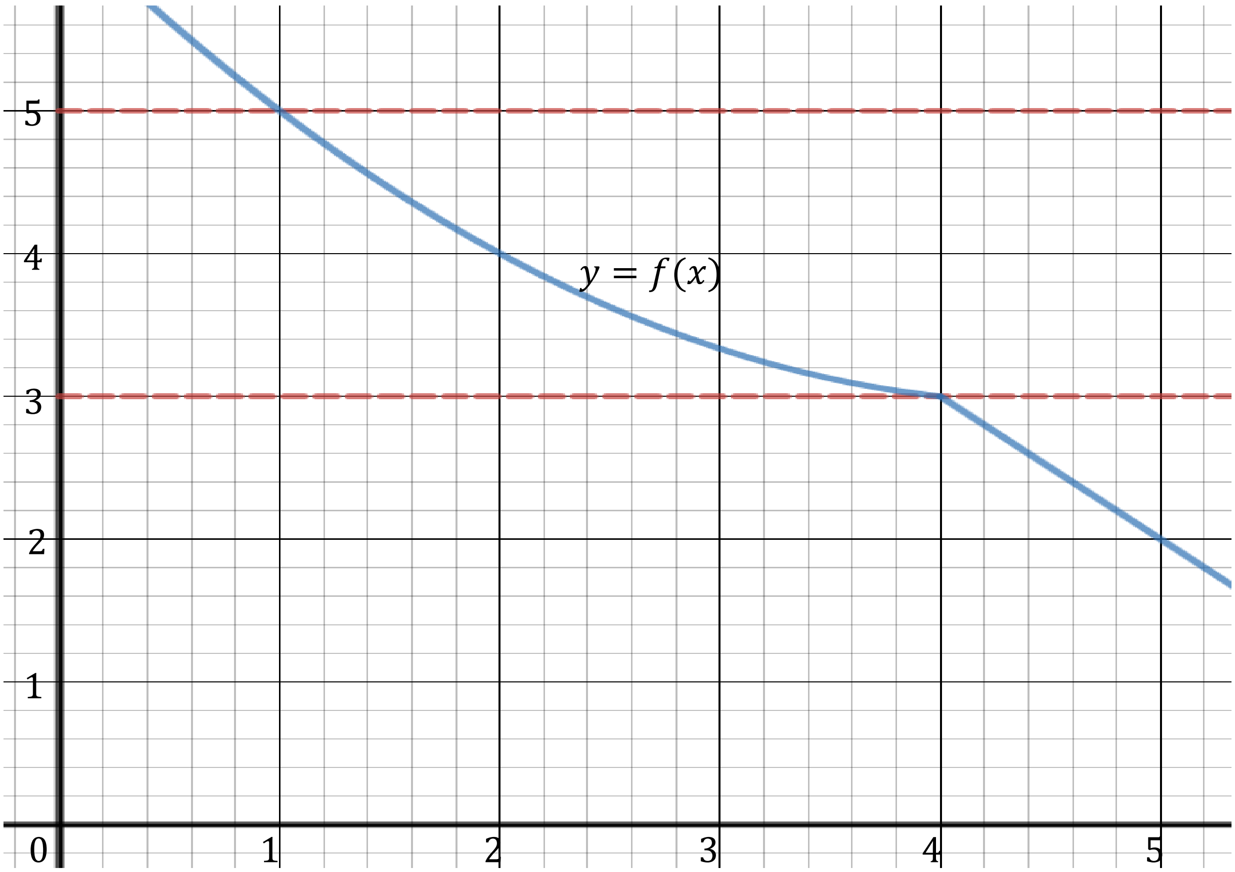
$$0 < |x - 3| < \delta \Longrightarrow \quad 3 - \delta < x < 3 + \delta.$$

We know $|f(x) - 5| < 3$ for all x where $1 < x < 6$.

So, $\quad 3 - \delta \geq 1 \quad$ and $\quad 3 + \delta \leq 6$

or: $\quad 2 \geq \delta \quad$ and $\quad \delta \leq 3 \quad \Longrightarrow \quad \delta \leq 2.$

Ex. For the function $f(x)$ whose graph is below, we have $\lim_{x \to 2} f(x) = 4$. Suppose that If $0 < |x - 2| < \delta$ then $|f(x) - 4| < 1$. Find the largest possible δ.



Notice that $3 < f(x) < 5$ when $1 < x < 4$.

So $|f(x) - 4| < 1$ when $1 < x < 4$.

$0 < |x - 2| < \delta \implies \quad 2 - \delta < x < 2 + \delta$.

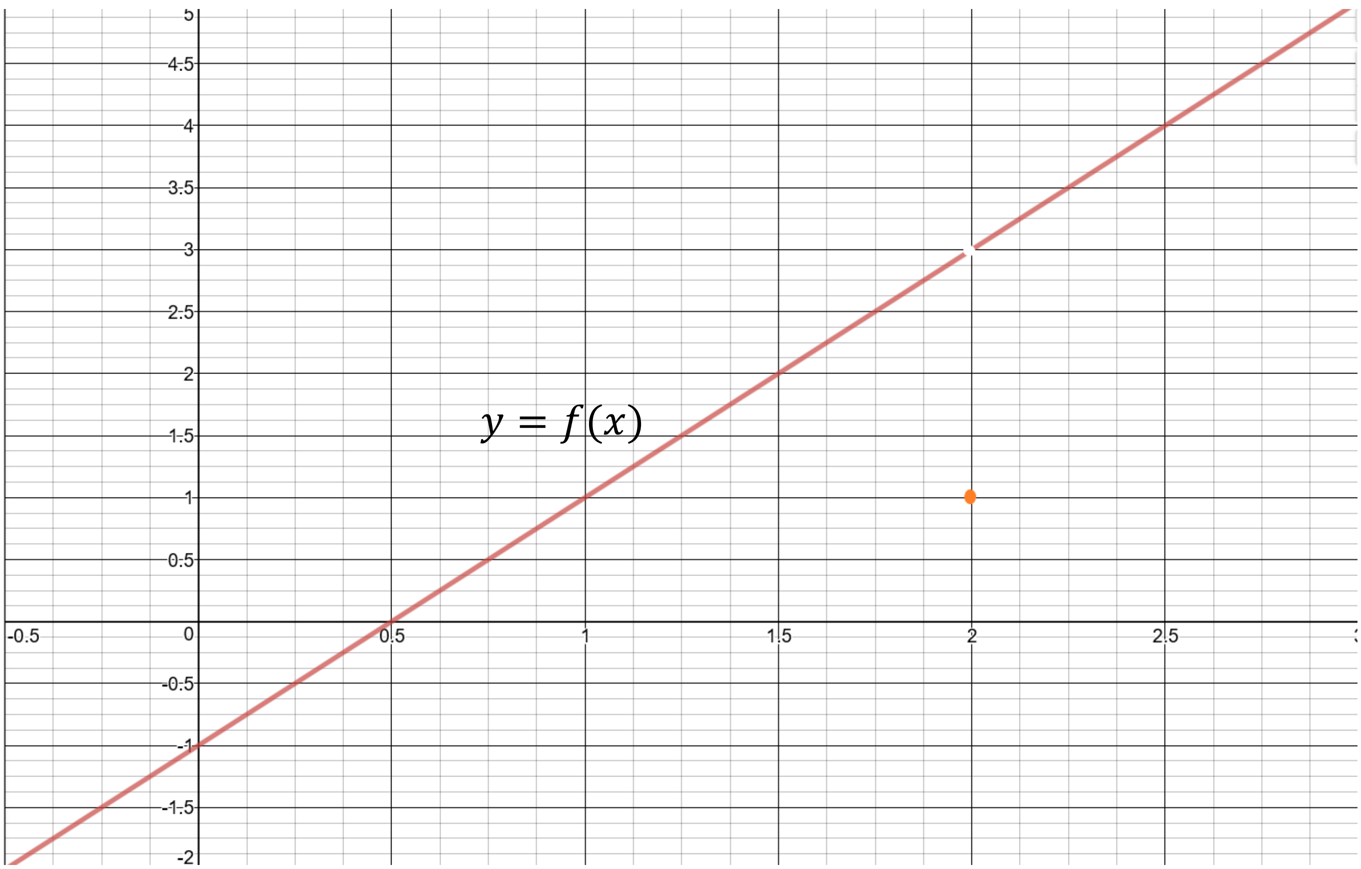
So $2 - \delta \geq 1$ and $2 + \delta \leq 4$ or

$1 \geq \delta$ and $\delta \leq 2 \implies \delta \leq 1$.

Ex. Suppose $f(x) = 2x - 1 \qquad if\ x \neq 2$

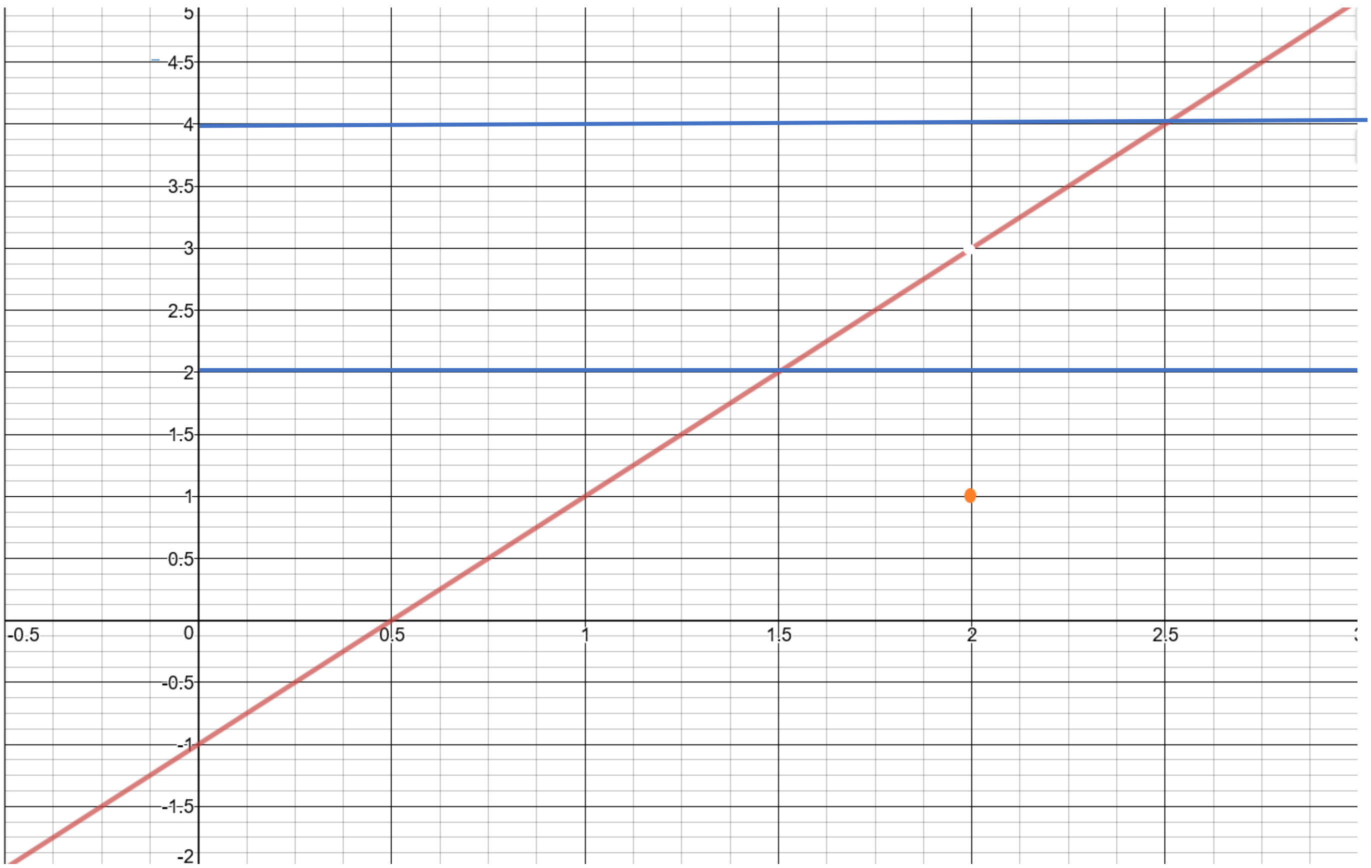
$\qquad\qquad\qquad\quad = 1 \qquad\qquad if\ x = 2.$

Prove that $\lim_{x\to 2} f(x) = 3.$



We must show that given any $\epsilon > 0$ there exists (or we can find) a $\delta > 0$ such that $|f(x) - 3| < \epsilon$ whenever $0 < |x - 2| < \delta$.

Let's start by drawing a picture when $\epsilon = 1$.



Notice that if $\epsilon = 1$, we need to find a δ such that if $2 - \delta < x < 2 + \delta,\ \ x \neq 2$, then $f(x)$ satisfies $L - \epsilon < f(x) < L + \epsilon$ or in this case,

$$3 - 1 < f(x) < 3 + 1 \qquad \text{i.e.,} \qquad 2 < f(x) < 4.$$

Since we don't care what happens to f at $x = 2$, this inequality is the same as

$$2 < 2x - 1 < 4\,.$$

Let's solve this inequality for x.

$$3 < 2x < 5$$

$$\frac{3}{2} < x < \frac{5}{2}\,.$$

So if $\frac{3}{2} < x < \frac{5}{2}$ then $2 < f(x) < 4$ or $|f(x) - 3| < \epsilon = 1$.

But what δ forces $\frac{3}{2} < x < \frac{5}{2}$?

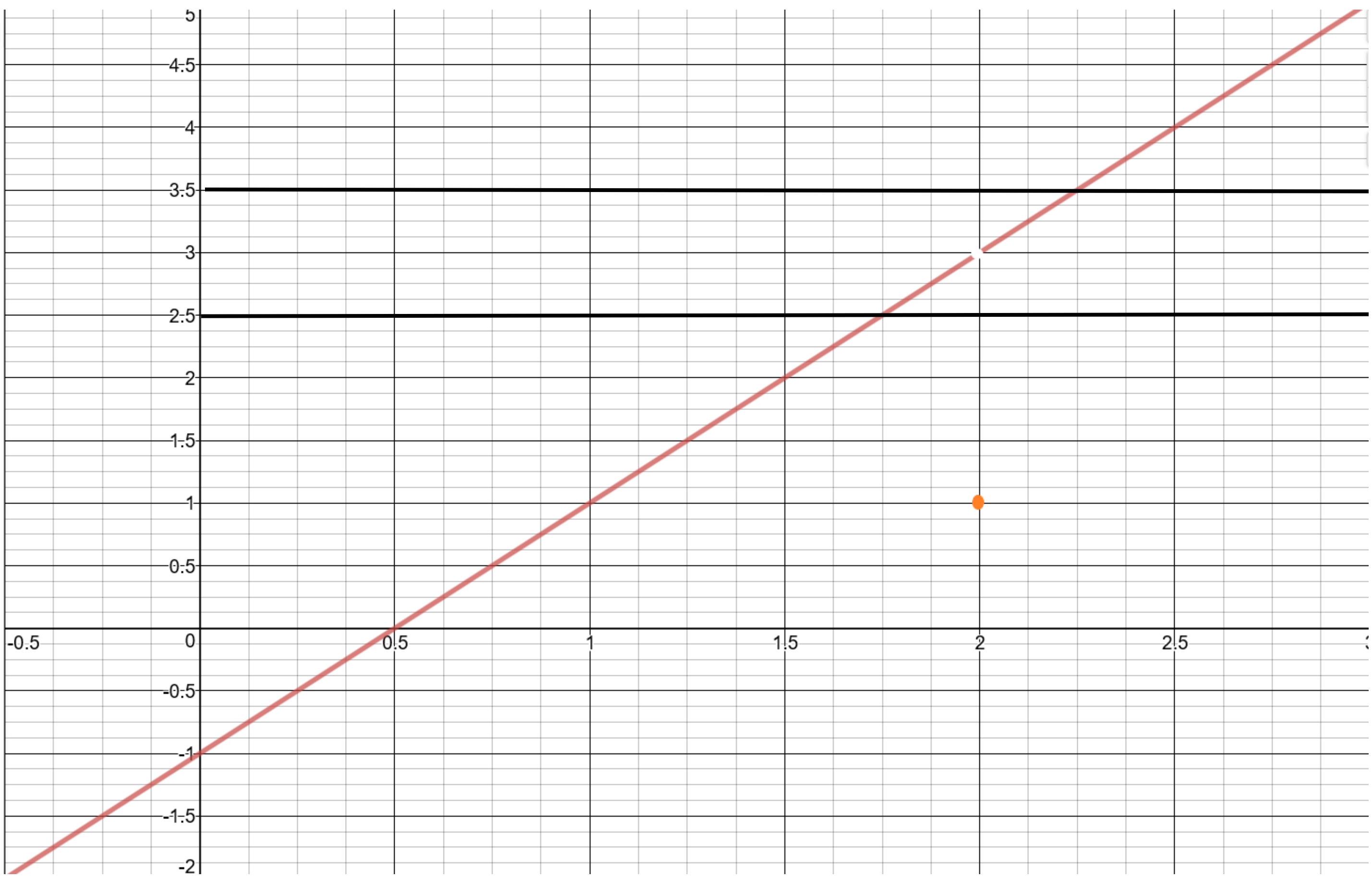
If we solve this inequality for $x - 2$ instead of x we will see the answer.

$$\frac{3}{2} - 2 < x - 2 < \frac{5}{2} - 2 \quad \text{or} \quad -\frac{1}{2} < x - 2 < \frac{1}{2}.$$

So if $\epsilon = 1$ then $\delta = \frac{1}{2}$ will force $|f(x) - 3| < \epsilon = 1$ to be true

whenever $0 < |x - 2| < \delta = \frac{1}{2}$.

But that only works if $\epsilon = 1$. What if $\epsilon = \frac{1}{2}$?



Now we need to find a δ such that if $2-\delta < x < 2+\delta, \;\; x \neq 2$

then $f(x)$ satisfies $L-\epsilon < f(x) < L+\epsilon$ or in this case,

$$3-\frac{1}{2} < f(x) < 3+\frac{1}{2}$$

i.e.,
$$\frac{5}{2} < f(x) < \frac{7}{2}.$$

now let's solve the inequality $\frac{5}{2} < 2x-1 < \frac{7}{2}$ for $x-2$:

$$\frac{5}{2} < 2x-1 < \frac{7}{2}$$

$$\frac{7}{2} < 2x < \frac{9}{2}$$

$$\frac{7}{4} < x < \frac{9}{4}$$

$$\frac{7}{4}-2 < x-2 < \frac{9}{4}-2$$

$$-\frac{1}{4} < x-2 < \frac{1}{4}.$$

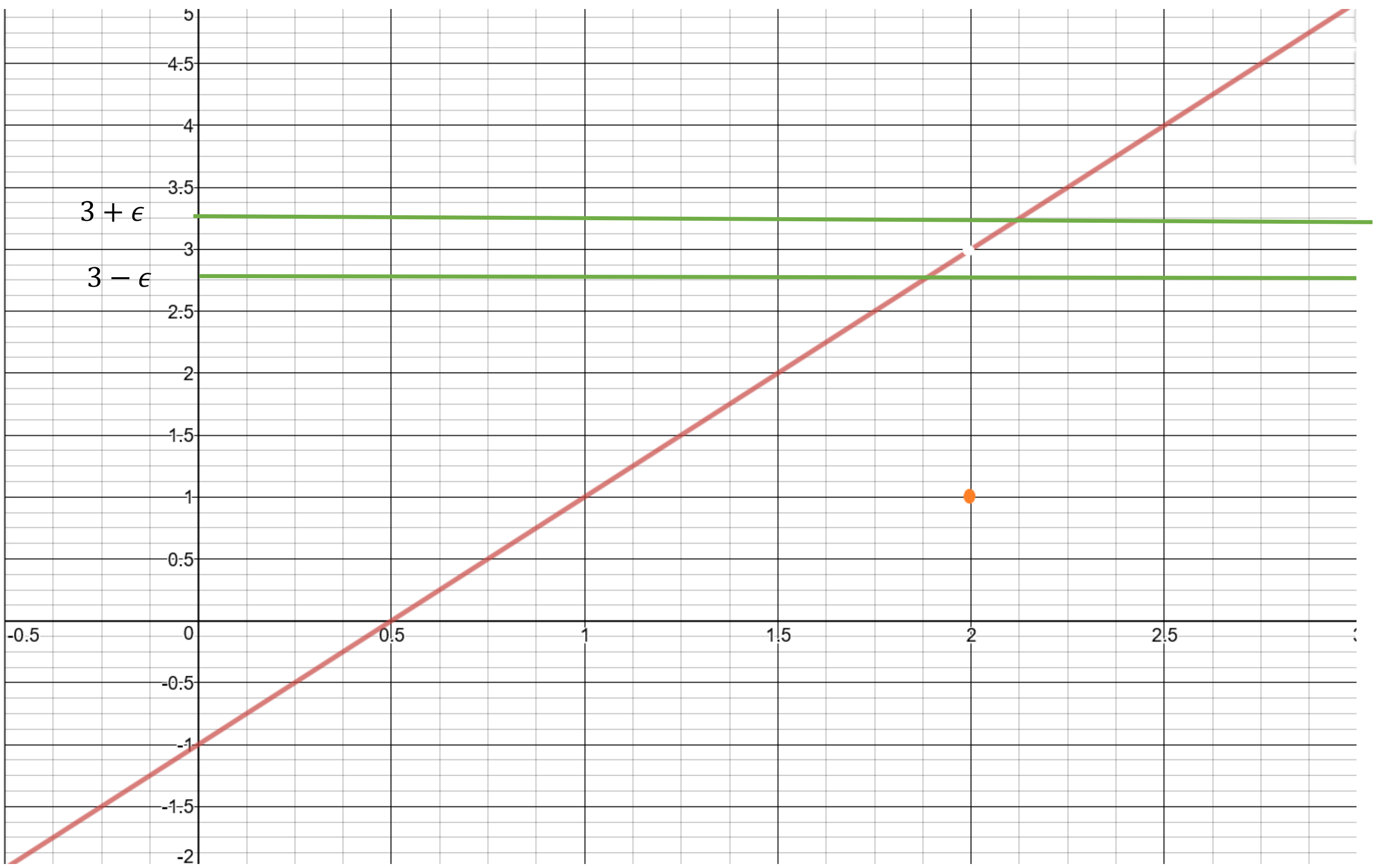
So if $\epsilon = \frac{1}{2}$ then $\delta = \frac{1}{4}$ will ensure that $|f(x)-3| < \epsilon = \frac{1}{2}$ to be true

whenever $0 < |x-2| < \delta = \frac{1}{4}$.

That' still not good enough. To prove $\lim_{x\to 2} f(x) = 3$ we must be able to find a δ for

ANY $\epsilon > 0$ no matter how small it is.

To find a δ for ANY $\epsilon > 0$ we need to go through the same procedure, but instead of a concrete ϵ (like 1 or $\frac{1}{2}$), we solve our inequalities for a general ϵ.



So we need to find a δ such that if $2 - \delta < x < 2 + \delta, \quad x \neq 2$

then $f(x)$ satisfies $L - \epsilon < f(x) < L + \epsilon$ or in this case,

$$3 - \epsilon < f(x) < 3 + \epsilon.$$

So let's solve $3 - \epsilon < 2x - 1 < 3 + \epsilon$ for $x - 2$ just as we did before.

$$3 - \epsilon < 2x - 1 < 3 + \epsilon$$
$$4 - \epsilon < 2x < 4 + \epsilon$$
$$\frac{4-\epsilon}{2} < x < \frac{4+\epsilon}{2}$$
$$2 - \frac{\epsilon}{2} < x < 2 + \frac{\epsilon}{2}$$
$$-\frac{\epsilon}{2} < x - 2 < \frac{\epsilon}{2}.$$

So if $\delta = \frac{\epsilon}{2}$ then $|f(x) - 3| < \epsilon$ whenever $0 < |x - 2| < \delta$.

Let's show that this δ works.

If $\delta = \frac{\epsilon}{2}$ then since $|x - 2| < \delta = \frac{\epsilon}{2}$

$$2|x - 2| < \epsilon$$
$$|2x - 4| < \epsilon$$
$$|(2x - 1) - 3| < \epsilon$$
$$|f(x) - 3| < \epsilon, \text{ since } f(x) = 2x - 1 \text{ when } x \neq 2.$$

Hence $\lim_{x \to 2} f(x) = 3$.

So the strategy to prove a limit is to start with the ϵ statement and work the inequality until you can get the δ inequality to pop out.

Notice that this example would have been essentially the same if we were trying to prove the $\lim_{x \to 2} f(x) = 3$ for $f(x) = 2x - 1$ or $f(x) = \frac{4x^2-1}{2x+1};\ \ x \neq -\frac{1}{2}$.