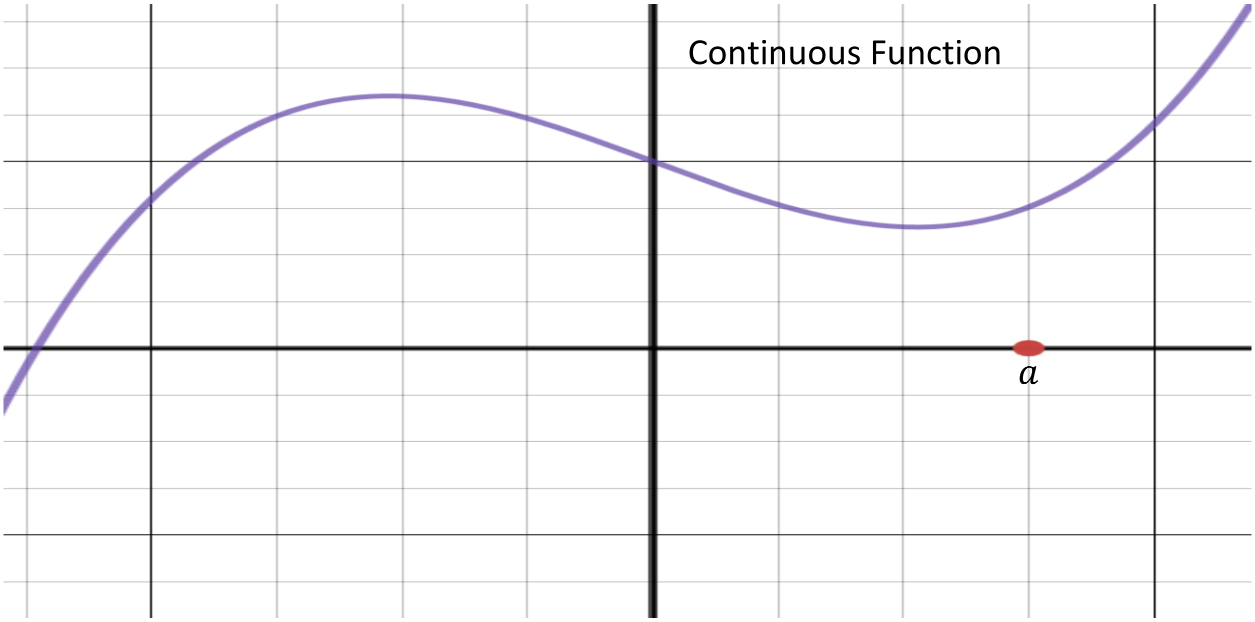
Continuity

Graphically, a function $f(x)$ is continuous at $x = a$ if you don't need to lift your pencil off the paper as you draw the graph of $f(x)$ around $x = a$.



Def. A function $f(x)$ is **continuous** at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

We need 3 things to occur for a function $f(x)$ be continuous at $x = a$:

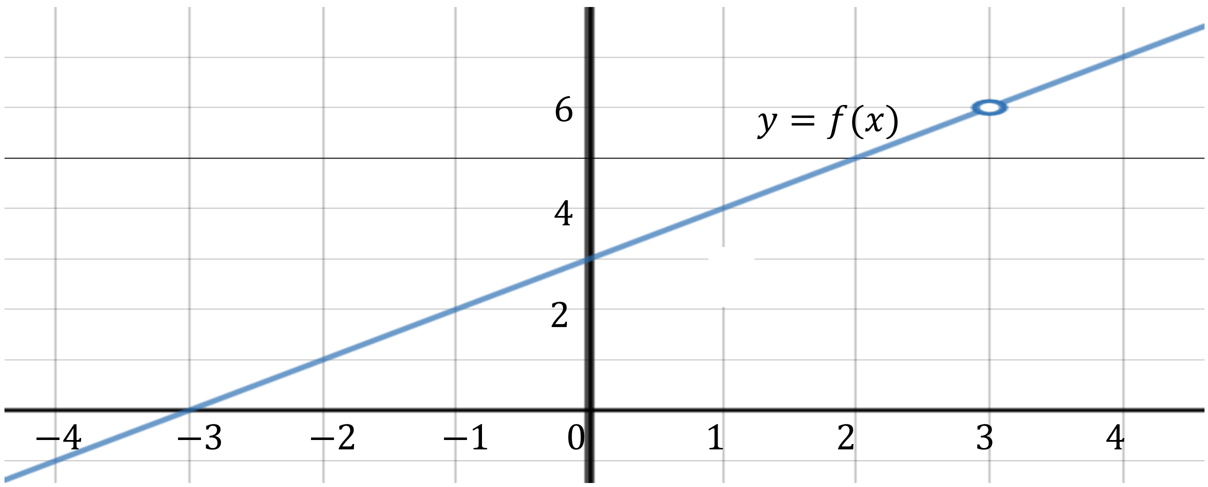
1. $f(x)$ is defined at $x = a$, i.e., a is in the domain of f
2. $\lim_{x \to a} f(x)$ exists (and is finite)
3. $\lim_{x \to a} f(x) = f(a)$.

$f(x)$ is said to be **discontinuous** at $x = a$ if f is not continuous at $x = a$.

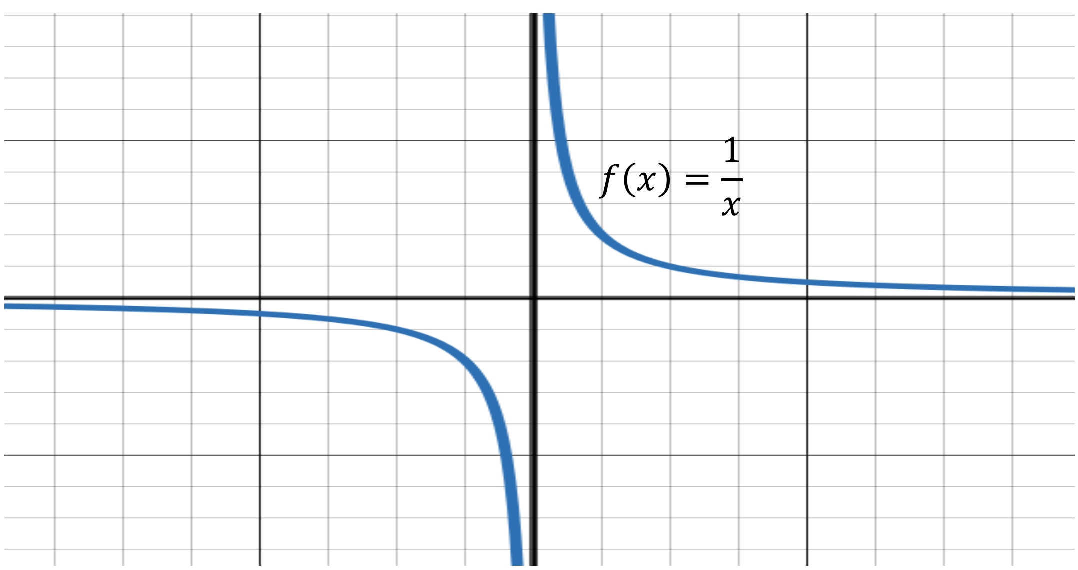
Ex. Points of discontinuity:

1. a. $f(x) = \frac{x^2-9}{x-3}$; $x \neq 3$; has a discontinuity at $x = 3$ because $f(x)$ is not defined there.

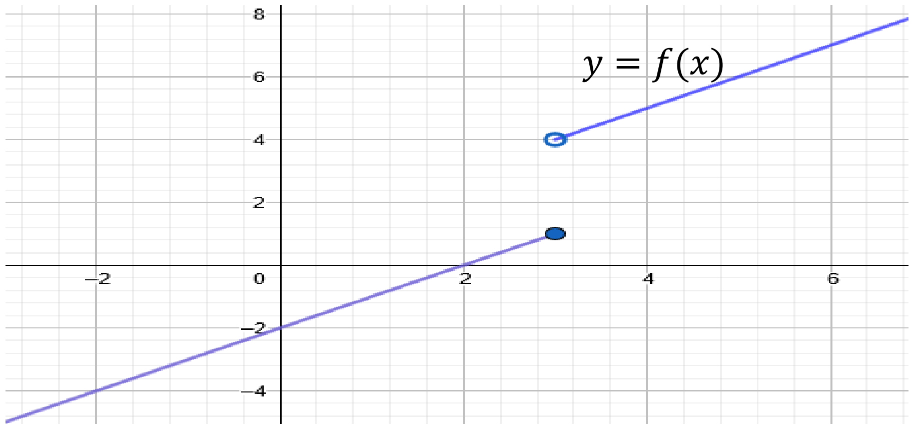
$$f(x) = \frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3} = x + 3; \quad x \neq 3.$$



b. $f(x) = \frac{1}{x}$; $x \neq 0$; has a discontinuity at $x = 0$ because $f(x)$ is not defined there (and because the $\lim_{x \to 0} f(x)$ doesn't exist).



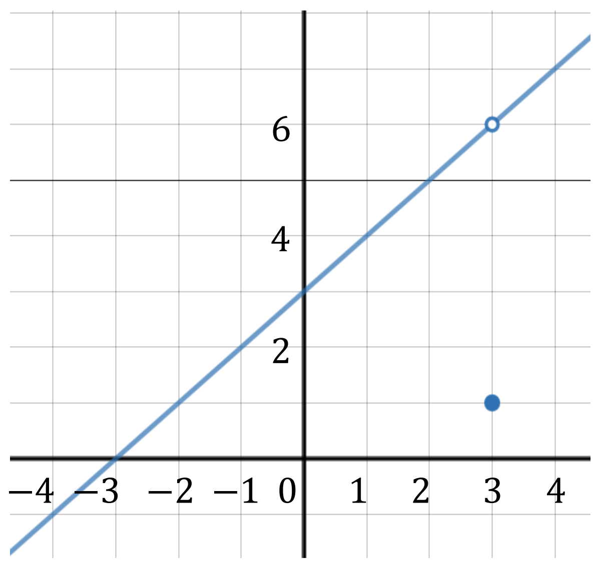
2. $f(x) = x + 1 \quad if\ x > 3$
$\quad\quad = x - 2 \quad if\ x \leq 3.$



Has a discontinuity at $x = 3$ because $\lim_{x \to 3} f(x)$ doesn't exist. This is called a **jump discontinuity**.

3. $f(x) = \frac{x^2-9}{x-3} \quad x \neq 3$
$\quad\quad = 1 \quad x = 3.$

Has a discontinuity at $x = 3$. The $\lim_{x \to 3} f(x)$ exists (what is it?) but it doesn't equal $f(3)$. This type of discontinuity is called a **removable discontinuity** because if we just redefine the function at $f(3)$ to be equal to $\lim_{x \to 3} f(x)$ the function would be continuous at $x = 3$.



Theorem: If f and g are continuous at $x = a$, and c is a real number, then the following functions are continuous at $x = a$.

1. $f + g$
2. $f - g$
3. cf
4. fg
5. f/g, provided $g(a) \neq 0$
6. $(f(x))^n$, where n is a positive integer.

These all follow from our limit rules.

For example: if $\lim_{x \to a} f(x) = f(a)$, and $\lim_{x \to a} g(x) = g(a)$, then

$$\begin{aligned} \lim_{x \to a}(f(x) + g(x)) &= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \\ &= f(a) + g(a). \end{aligned}$$

We also have as a result of our limit rules that:

1. **All polynomials are continuous for all values of x**
2. **All rational functions are continuous for all values of x in their domain.**

Ex. For what values of x is $f(x) = \frac{2x-5}{x^2-2x-8}$ continuous?

$f(x) = \frac{2x-5}{x^2-2x-8} = \frac{2x-5}{(x-4)(x+2)}$; so only $x = 4, -2$ are not in the domain of f.

Thus, since $f(x)$ is a rational function, it is continuous for all x such that $x \neq 4, -2$.

In other words, $f(x)$ is discontinuous only at $x = 4, -2$.

Theorem (Continuity of Composite functions at $x = a$):

If g is continuous at $x = a$ and f is continuous at $g(a)$, then the composite function $f(g(x))$ is continuous at $x = a$.

Limits of Composite Functions:

If $\lim_{x \to a} g(x) = L$ and f is continuous at L, then

$\lim_{x \to a} f\big(g(x)\big) = f(\lim_{x \to a} g(x))$.

Note: if $g(x)$ is continuous at $x = a$ then

$$\lim_{x \to a} f\big(g(x)\big) = f(\lim_{x \to a} g(x)).$$

Ex. Evaluate

a. $\lim_{x\to -2} \sqrt{2x^2+3}$
b. $\lim_{x\to -3} \sin\left(\frac{x^2-9}{x+3}\right)$.

We will see later that both $\sqrt{x}$ (for $x > 0$) and $sinx$ (for all x) are continuous (in fact all 6 trig functions are continuous in their domains).

a. We can think of $\sqrt{2x^2+3}$ as a composition of $f(x) = \sqrt{x}$ and $g(x) = 2x^2+3$; $\quad f\big(g(x)\big) = \sqrt{2x^2+3}$. Thus

$$\lim_{x\to -2} \sqrt{2x^2+3} = \sqrt{\lim_{x\to -2}(2x^2+3)} = \sqrt{(8+3)} = \sqrt{11}.$$

b. $$\lim_{x\to -3} \sin\left(\frac{x^2-9}{x+3}\right) = sin\ (\lim_{x\to -3} \frac{x^2-9}{x+3})$$
$$= \sin\ (\lim_{x\to -3} \frac{(x+3)(x-3)}{x+3})$$
$$= sin\ (\lim_{x\to -3}(x-3)) = \sin(-6).$$

Notice that the inner function, $g(x) = \frac{x^2-9}{x+3}$, is not continuous at $x = -3$, but it does have a limit at $x = -3$, which is all we need to use the previous theorem on limits of composite functions.

Def. A function is **continuous from the right at $x = a$** if $\lim_{x \to a^+} f(x) = f(a)$ and **continuous from the left at $x = a$** if $\lim_{x \to a^-} f(x) = f(a)$.

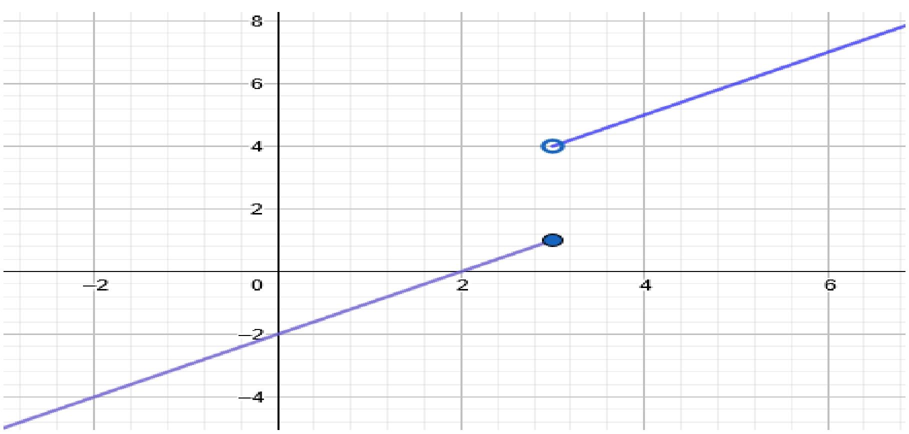
Notice that f is continuous at $x = a$ if and only if it's continuous from the right and continuous from the left at $x = a$.

Def. A function f is **continuous on an interval I** if it is continuous at all points of I. If I contains its endpoints, continuity on I means continuous from the right or left at the relevant endpoints.

Ex. Determine the intervals of continuity for

$$f(x) = x + 1 \quad if\ x > 3$$
$$= x - 2 \quad if\ x \le 3.$$

Start by sketching the graph of f.



If $x < 3$ then $f(x) = x - 2$ and for $x = a < 3$, $\lim_{x \to a} f(x) = f(a)$

(f is a polynomial for $x < 3$ and a different polynomial for $x > 3$.)

If $x > 3$ then $f(x) = x + 1$ and for $x = a > 3$, $\lim_{x \to a} f(x) = f(a)$.

The only question is at $x = 3$, where $f(3) = 1$.

Notice that: $\lim_{x\to 3^-} f(x) = 1$ and $\lim_{x\to 3^+} f(x) = 4$ so $\lim_{x\to 3} f(x) = DNE$.

So f is continuous on: $(-\infty, 3) \cup (3, \infty)$.

Continuity of Functions involving Roots

Recall that Limit Law #7 said:

$\lim_{x\to a}(f(x))^{\frac{n}{m}} = (\lim_{x\to a} f(x))^{\frac{n}{m}}$; provided $f(x) > 0$,

for x near a, if m is even and n/m is reduced to lowest form and $m, n > 0$.

So if $f(x)$ is a continuous function (i.e. $\lim_{x\to a} f(x) = f(a)$) we have:

$$\lim_{x\to a}(f(x))^{\frac{n}{m}} = (\lim_{x\to a} f(x))^{\frac{n}{m}} = (f(a))^{\frac{n}{m}}.$$

In other words, assume m and n are positive integers with no common factors. If m is odd then $(f(x))^{\frac{n}{m}}$ is continuous at all points at which f is continuous. If m is even, then $(f(x))^{\frac{n}{m}}$ is continuous at all point $x = a$ at which f is continuous and $f(a) > 0$.

Ex. For what values of x are the following functions continuous?

a. $h(x) = \sqrt{25 - x^2}$

b. $g(x) = (x^2 + 3x - 6)^{\frac{4}{7}}$

a. $h(x) = \sqrt{25 - x^2} = (25 - x^2)^{\frac{1}{2}}$, in this case $h(x) = (f(x))^{\frac{1}{2}}$,

where $f(x) = 25 - x^2$.

Since $m = 2$ is even, $h(x)$ will be continuous when $f(x) > 0$, i.e.,

$$-5 < x < 5.$$

Now we need to check continuity at the endpoints, $x = -5, 5$.

$$\lim_{x \to -5^+} \sqrt{25 - x^2} = 0 = h(-5)$$

$$\lim_{x \to 5^-} \sqrt{25 - x^2} = 0 = h(5).$$

So $h(x) = \sqrt{25 - x^2}$ is continuous on $[-5,5]$.

b. $g(x) = (x^2 + 3x - 6)^{\frac{4}{7}}$; $m = 7$ is odd so $g(x)$ is continuous everywhere since $f(x) = x^2 + 3x - 6$ is continuous everywhere. So $g(x)$ is cont. on $(-\infty, \infty)$.

Continuity of Trig Functions

Earlier we used the squeeze theorem to show that:

$\lim_{h\to 0}\sin(h)=0$ and $\lim_{h\to 0}\cos(h)=1.$

Now we want to show that $sinx$ and $cosx$ are continuous functions for any $x=a$.

So we must show:

$$\lim_{x\to a}\sin(x)=\sin(a) \quad \text{and} \quad \lim_{x\to a}\cos(x)=\cos(a).$$

To show $\lim_{x\to a}\sin(x)=\sin(a)$ remember that

$$\sin(a+h)=(sina)(\cos(\mathrm{h}))+(\sin(\mathrm{h}))(\cos(\mathrm{a})).$$

If we let $x=a+h$ then

$$\lim_{x\to a}\sin(x)=\lim_{h\to 0}\sin(a+h)$$
$$=\lim_{h\to 0}[(sina)(\cos(\mathrm{h}))+(\sin(\mathrm{h}))(\cos(\mathrm{a}))]$$
$$=\lim_{h\to 0}(\sin(a))(\cos(h))+\lim_{h\to 0}(\sin(h))(\cos(a))$$
$$=\sin(a)+0=\sin(a).$$

To show that $cosx$ is continuous for any $x=a$ we do the same trick, but we use

$$\cos(a+h)=(\cos(a))(\cos(h))-(\sin(a))(\sin(h)).$$

Using the fact that if f and g are continuous at $x=a$, so is f/g as long as $g(a)\neq 0$, we can conclude:

The 6 trig functions are continuous at all points of their domains.

Ex. Evaluate $\lim_{x \to \frac{\pi}{2}} \sqrt{\frac{4sin^2x-4}{sinx-1}}$.

$$\lim_{x \to \frac{\pi}{2}} \sqrt{\frac{4sin^2x-4}{sinx-1}} = \lim_{x \to \frac{\pi}{2}} \sqrt{\frac{4(sin^2x-1)}{sinx-1}}$$

$$= \lim_{x \to \frac{\pi}{2}} 2\sqrt{\frac{(sinx-1)(sinx+1)}{sinx-1}}$$

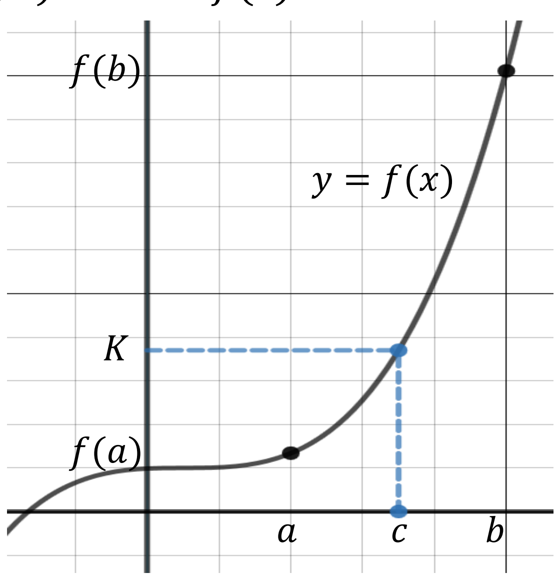
$$= \lim_{x \to \frac{\pi}{2}} 2\sqrt{sinx+1} = 2\sqrt{1+1} = 2\sqrt{2}.$$

The Intermediate Value Theorem

Frequently, we want to know if there is a solution to the problem $f(x) = K$. That is, is there a real number x_0 such that $f(x_0) = K$?

The Intermediate Value Theorem is a common way to show that an equation has a solution (without necessarily finding it).

Intermediate Value Theorem: Suppose f is continuous on the interval $[a, b]$ and K is a number strictly between $f(a)$ and $f(b)$. Then there exists at least one number c in (a, b) such that $f(c) = K$.



Ex. Show that the equation $sinx + x = 1$ has at least one solution on $(0, \frac{\pi}{2})$.

Let $f(x) = sinx + x - 1$.

$f(x)$ is a continuous function on $[0, \frac{\pi}{2}]$ and

$$f(0) = -1, \qquad f\left(\frac{\pi}{2}\right) = sin\frac{\pi}{2} + \frac{\pi}{2} - 1 = \frac{\pi}{2}.$$

Since $f(0) < 0$ and $f\left(\frac{\pi}{2}\right) > 0$,

by the intermediate value theorem there must be at least one point c,

where $0 < c < \frac{\pi}{2}$ such that $f(c) = 0$. That c is a solution of

$sinx + x = 1$.

Ex. Show that $\sqrt[3]{x+2} - x = 3$ has at least one solution in $[-10,6]$.

Let $f(x) = \sqrt[3]{x+2} - x - 3$.

Notice that $f(x)$ is continuous everywhere because $g(x) = \sqrt[3]{x+2}$ and

$h(x) = -x - 3$ are both continuous everywhere. So, in particular,

$f(x)$ is continuous on $[-10,6]$.

$$f(-10) = \sqrt[3]{-8} + 10 - 3 = -2 + 10 - 3 = 5 > 0$$

$$f(6) = \sqrt[3]{8} - 6 - 3 = 2 - 6 - 3 = -7 < 0.$$

So by the intermediate value theorem there must be at least one point c,

where $-10 < c < 6$ such that $f(c) = 0$. That c is a solution of

$\sqrt[3]{x+2} - x = 3$.