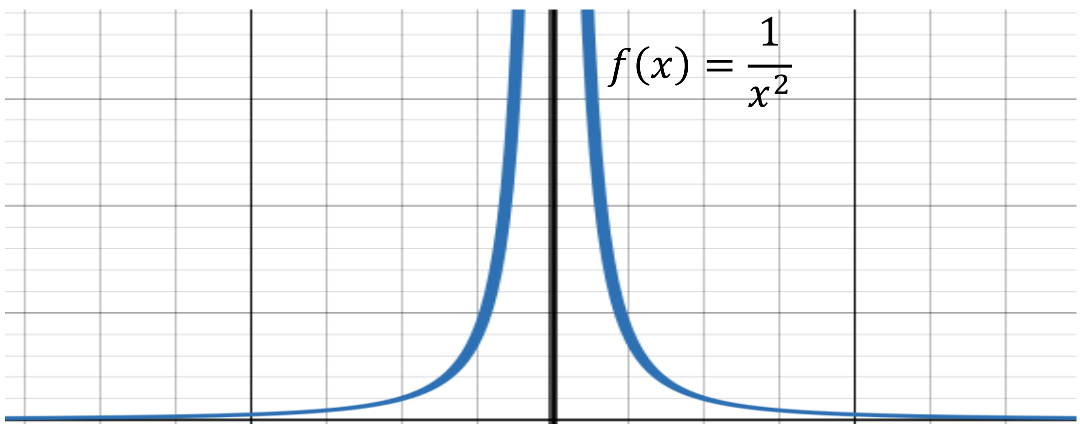
Infinite Limits

There are 2 types of limits that involve infinity that come up frequently.

1. An Infinite Limit- The value of a function increases or decreases without bound as x approaches a finite point a.
2. Limit at Infinity- finding a limit when x (or the independent variable) increases or decreases without bound (discussed in the next section)

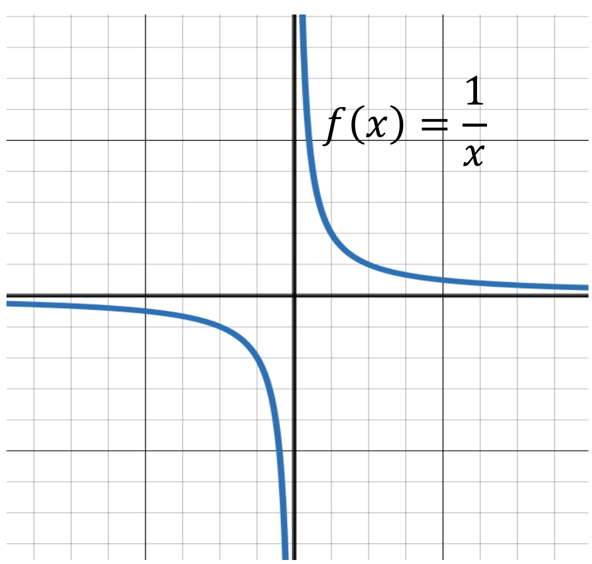
Ex. An Infinite Limit. Let $f(x) = \frac{1}{x^2}$. $\lim_{x \to 0} f(x) = \infty$.



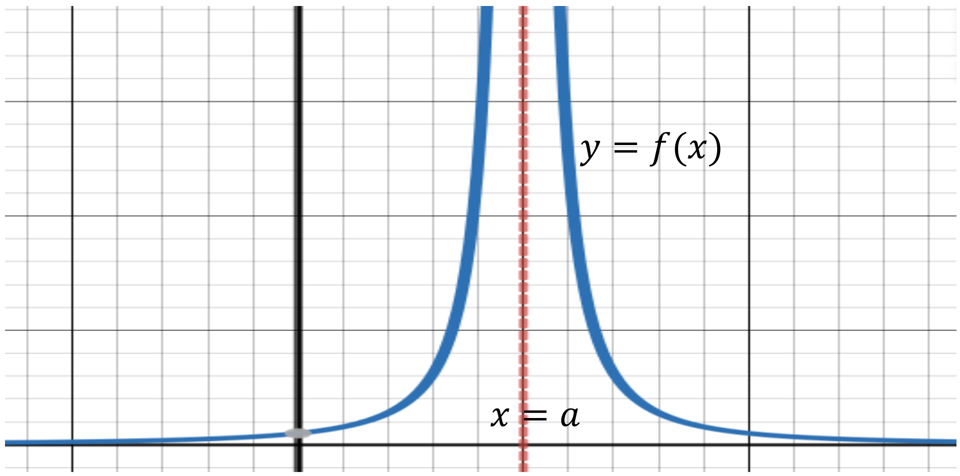
Ex. Let $f(x) = \frac{1}{x}$, Find $\lim_{x \to 0} f(x)$, if it exists. Start by graphing $f(x) = \frac{1}{x}$.

$\lim_{x \to 0} f(x) =$ Does Not Exist (DNE),

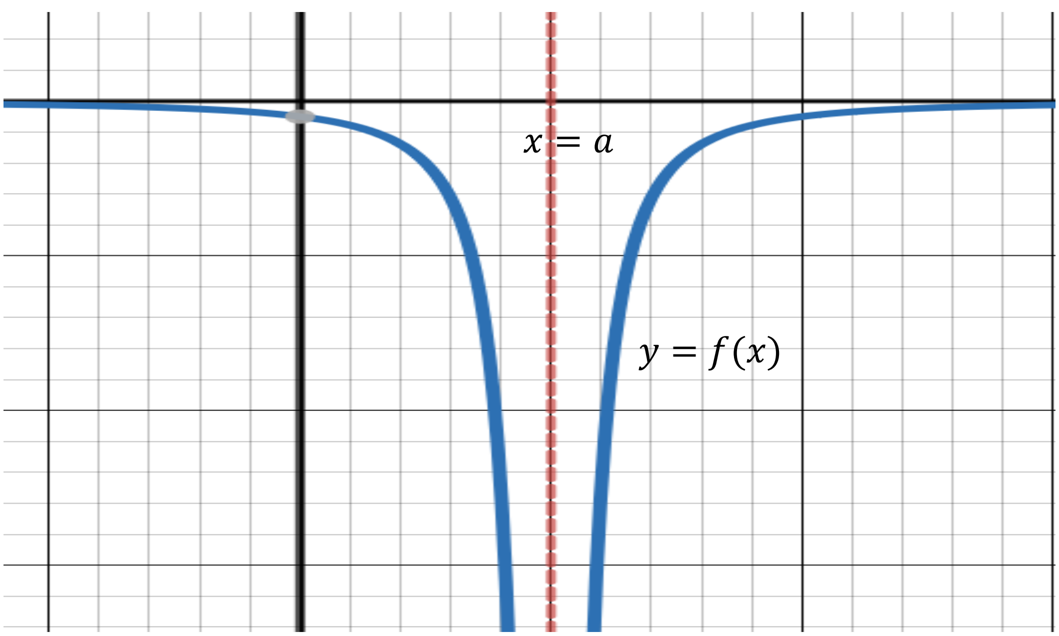
Because $f(x)$ approaches $+\infty$ from the right and $-\infty$ from the left.



Def. Suppose $f(x)$ is defined for all x near $x = a$. If $f(x)$ grows arbitrarily large for all x sufficiently close (but not equal) to a, we write $\lim_{x \to a} f(x) = \infty$ and say that the limit of $f(x)$ as x approaches a is infinity.



If $f(x)$ is negative and grows arbitrarily large in magnitude for all x sufficiently close (but not equal) to a, we write $\lim_{x \to a} f(x) = -\infty$ and say that the limit of $f(x)$ as x approaches a is negative infinity.

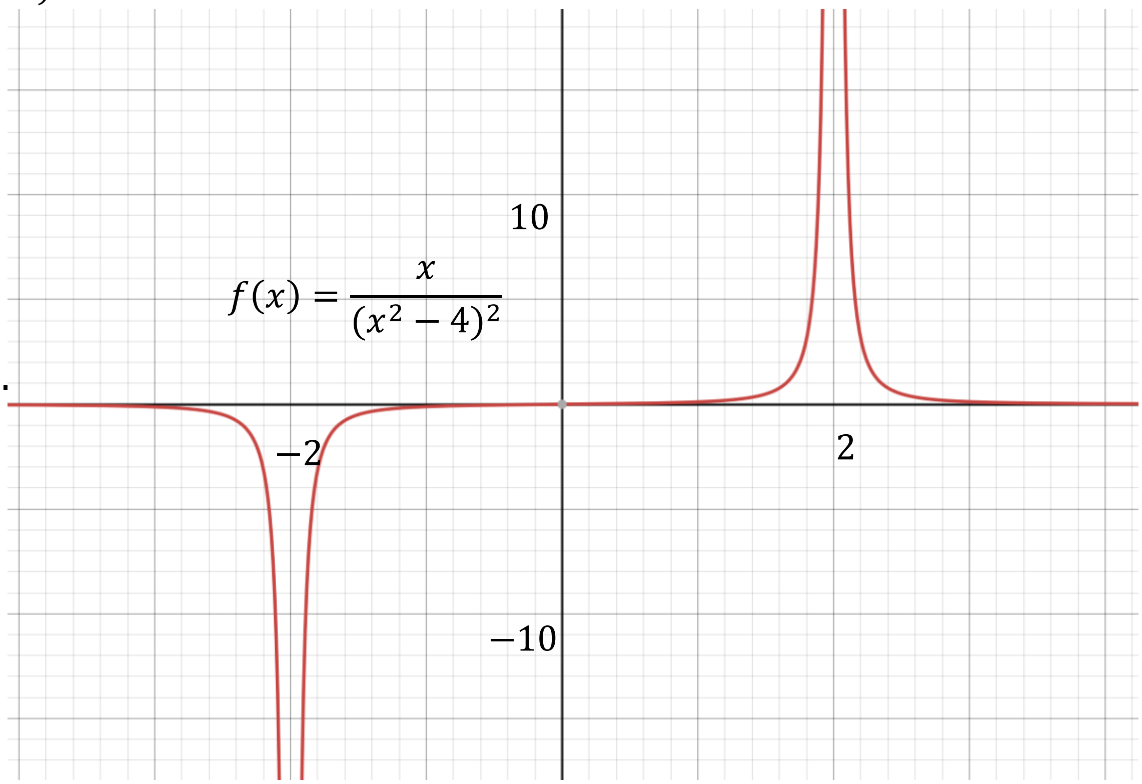


Ex. Find $\lim_{x \to 2} \frac{x}{(x^2-4)^2}$ and $\lim_{x \to -2} \frac{x}{(x^2-4)^2}$ given the graph of

$f(x) = \frac{x}{(x^2-4)^2}$.

$\lim_{x \to 2} \frac{x}{(x^2-4)^2} = \infty$

$\lim_{x \to -2} \frac{x}{(x^2-4)^2} = -\infty$.



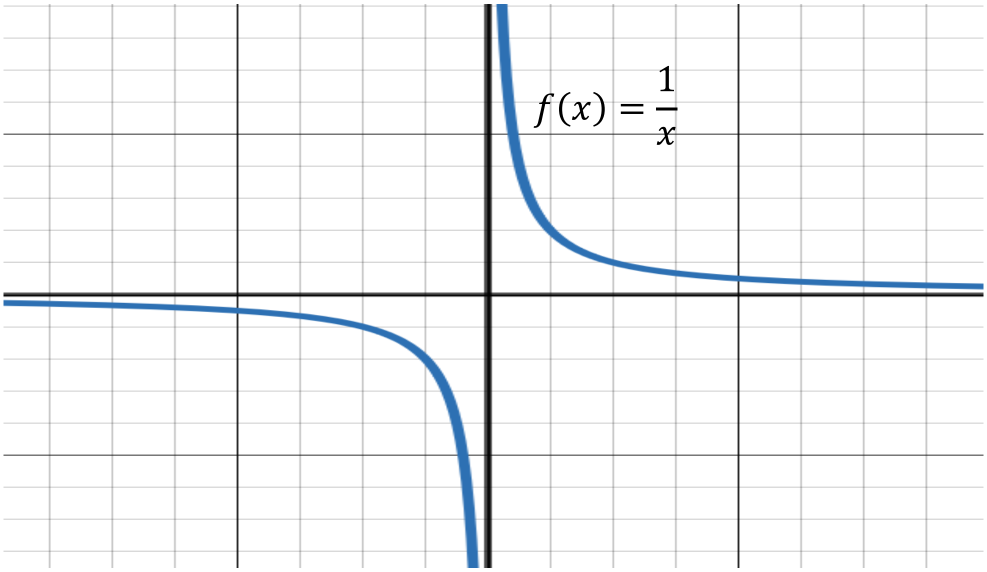
<u>One-Sided Limits</u>

Def. Suppose f is defined for all x near a with $x > a$. If $f(x)$ becomes arbitrarily large for all x sufficiently close to a with $x > a$, we write $\lim_{x \to a^+} f(x) = \infty$. If $f(x)$ is negative and grows arbitrarily large in magnitude for all x sufficiently close to a with $x > a$, we write $\lim_{x \to a^+} f(x) = -\infty$.

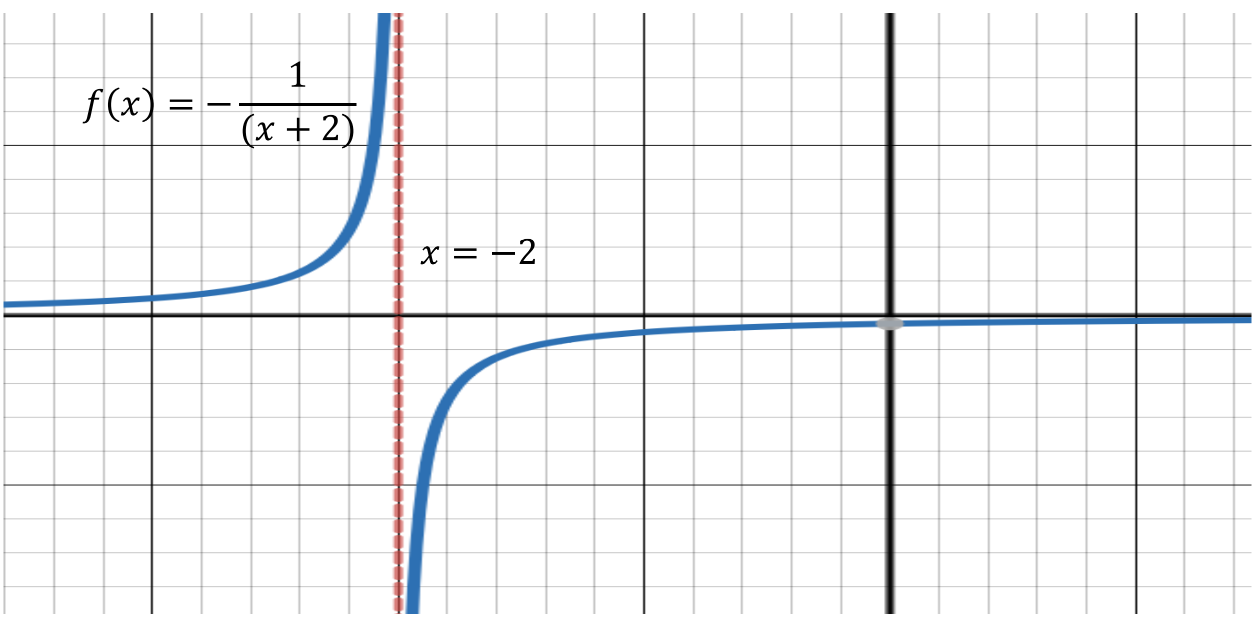
Suppose f is defined for all x near a with $x < a$. If $f(x)$ becomes arbitrarily large for all x sufficiently close to a with $x < a$, we write $\lim_{x \to a^-} f(x) = \infty$.

If $f(x)$ is negative and grows arbitrarily large in magnitude for all x sufficiently close to a with $x < a$, we write $\lim_{x \to a^-} f(x) = -\infty$.

Ex. Let $f(x) = \frac{1}{x}$, then $\lim_{x \to 0^+} f(x) = \infty$ and $\lim_{x \to 0^-} f(x) = -\infty$.



Ex. Let $f(x) = -\frac{1}{(x+2)}$, then $\lim_{x \to -2^+} f(x) = -\infty$ and $\lim_{x \to -2^-} f(x) = \infty$



Def. If $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a^+} f(x) = \pm\infty$, or

$\lim_{x\to a^-} f(x) = \pm\infty$, the line $x = a$ is called a **vertical asymptote**.

Ex. find $\lim_{x\to 3^-} \frac{2x}{x-3}$, $\lim_{x\to 3^+} \frac{2x}{x-3}$, and $\lim_{x\to 3} \frac{2x}{x-3}$ if they exist. Find any vertical asymptotes.

Notice that as x goes toward $x = 3$, $\frac{2x}{x-3}$ is going to go toward either $\pm\infty$ (why?). The only question is as x approaches 3 from the left or the right, is the function approaching ∞ or $-\infty$?

To answer this we just need to know the sign of $\frac{2x}{x-3}$ as we approach $x = 3$ from either side.

Notice when x is slightly larger than 3, $2x$ is a positive number, and $x - 3$ is a positive number. Thus for x slightly larger than 3 (ie, we approach 3 from the right) $\frac{2x}{x-3}$ is a ratio of positive numbers and hence positive. Thus

$\lim_{x\to 3^+} \frac{2x}{x-3} = \infty.$

When x is slightly smaller than $3, 2x$ is still positive, but $x - 3$ is negative. Thus $\frac{2x}{x-3}$ is a ratio of a positive number and a negative number and hence negative. Thus $\lim_{x\to 3^-} \frac{2x}{x-3} = -\infty.$

Since $\lim_{x\to 3^+} \frac{2x}{x-3} = \infty$ (or we could have used $\lim_{x\to 3^-} \frac{2x}{x-3} = -\infty$), $x = 3$ is a vertical asymptote for the graph of $f(x) = \frac{2x}{x-3}$

since $\lim_{x\to 3^-} \frac{2x}{x-3} \neq \lim_{x\to 3^+} \frac{2x}{x-3}$, $\lim_{x\to 3} \frac{2x}{x-3}$ does not exist.

Ex. Find $\lim_{x\to 1^-} \frac{(2-x)(x+1)}{(x-1)^2}$, $\lim_{x\to 1^+} \frac{(2-x)(x+1)}{(x-1)^2}$, and $\lim_{x\to 1} \frac{(2-x)(x+1)}{(x-1)^2}$ if they exist. Find any vertical asymptotes.

If a rational function is not already factored (this one is), factor it.

Notice as x goes toward $x = 1$, $\frac{(2-x)(x+1)}{(x-1)^2}$ is going to go toward either $\pm\infty$.

If x is slightly larger than 1, then $2 - x$ is positive , $x + 1$ is positive , and $(x-1)^2$ is positive .

Hence $\frac{(2-x)(x+1)}{(x-1)^2}$ is positive.

So $\lim_{x\to 1^+} \frac{(2-x)(x+1)}{(x-1)^2} = +\infty$.

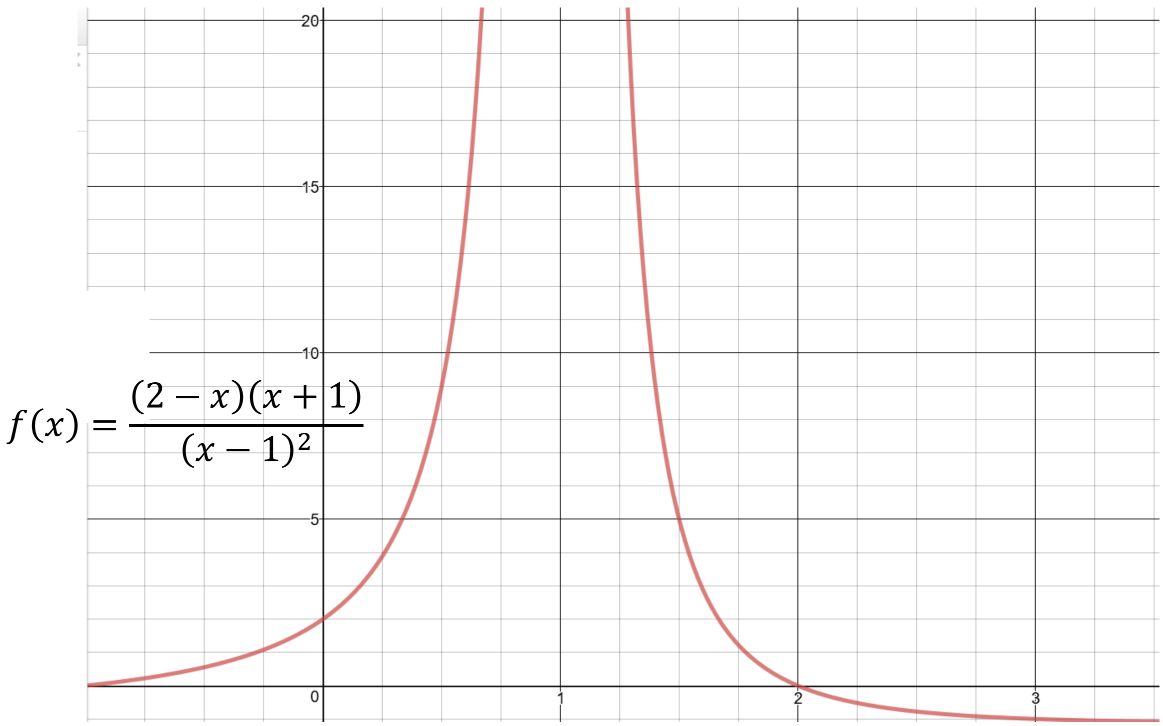
If x is slightly smaller than 1, then $2 - x$ is positive, $x + 1$ is positive, and $(x-1)^2$ is positive.

Hence $\frac{(2-x)(x+1)}{(x-1)^2}$ is positive.

So $\lim_{x\to 1^-} \frac{(2-x)(x+1)}{(x-1)^2} = +\infty$.

Thus $\lim_{x\to 1} \frac{(2-x)(x+1)}{(x-1)^2} = +\infty$

The graph of $f(x) = \frac{(2-x)(x+1)}{(x-1)^2}$ has a vertical asymptote at $x = 1$.



Ex. Find all vertical asymptotes of $f(x) = \frac{x^2-5x+6}{x^2-x-6}$. Evaluate $\lim_{x\to -2^+} f(x)$, $\lim_{x\to -2^-} f(x)$, and $\lim_{x\to -2} f(x)$ if they exist.

$$f(x) = \frac{x^2-5x+6}{x^2-x-6} = \frac{(x-3)(x-2)}{(x-3)(x+2)} = \frac{x-2}{x+2}; \quad \text{if } x \neq -2,3.$$

For x slightly larger than -2, $x-2$ is negative, $x+2$ is positive.

So $\frac{x-2}{x+2}$ is negative.

Thus $\lim_{x\to -2^+} f(x) = -\infty$.

For x slightly smaller than -2, $x-2$ is negative, $x+2$ is negative.

Thus $\frac{x-2}{x+2}$ is positive and $\lim_{x\to -2^-} f(x) = +\infty$.

$\lim_{x \to -2} f(x)$ =DNE because $\lim_{x \to -2^+} f(x) \neq \lim_{x \to -2^-} f(x)$.

$f(x) = \frac{x^2-5x+6}{x^2-x-6}$ has a vertical asymptote at $x = -2$.

Does $f(x)$ have an asymptote at $x = 3$? NO, since $\lim_{x \to 3} f(x) = \frac{1}{5}$.

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \frac{(x-3)(x-2)}{(x-3)(x+2)} = \lim_{x \to 3^+} \frac{x-2}{x+2} = \frac{1}{5} = \lim_{x \to 3^-} \frac{x-2}{x+2}.$$

Limits of Trig Functions

Ex. Find $\lim_{x \to 0^+} cotx$ and $\lim_{x \to 0^-} cotx$.

$cotx = \frac{cosx}{sinx}$ so $cotx$ goes to either $\pm\infty$ as x goes to zero because as x goes to zero $sinx$ goes to 0 and $cosx$ doesn't go to zero.

For x slightly larger than 0, $cosx$ is positive, $sinx$ is positive, so $cotx$ is positive.

Hence $\lim_{x \to 0^+} cotx = +\infty$

For x slightly smaller than 0, $cosx$ is positive, $sinx$ is negative, so $cotx$ is negative.

Hence $\lim_{x \to 0^-} cotx = -\infty$.

Does $cotx$ have an asymptote at $x = 0$? Yes, since $\lim_{x \to 0^+} cotx = +\infty$ (or $\lim_{x \to 0^-} cotx = -\infty$).