Net Change: Integrating the Derivative

If $s(t)$ is the position of an object moving along a line, then $s(b) - s(a)$ is the displacement of the object for $a \le t \le b$ (displacement to the right/up is positive, to the left/down is negative).

Recall that the velocity at time t is $v(t) = s'(t)$. Thus we have:

$$\int_a^b v(t)dt = \int_a^b s'(t)dt = s(b) - s(a) = \boldsymbol{displacement}$$

$$= \boldsymbol{Net\ Change\ in\ Position\ over}\ [\boldsymbol{a}, \boldsymbol{b}].$$

Displacement can be positive or negative. Distance is always non-negative. To find the distance travelled we need to integrate the speed $|v(t)|$.

$$\int_a^b |v(t)|dt = \boldsymbol{distance\ traveled\ for\ a \le t \le b}.$$

Ex. A particle moves along a straight line so that its velocity is

$$v(t) = t^2 - t - 6\ m/sec$$

a. Find the displacement during $1 \le t \le 4$.
b. Find the distance traveled during $1 \le t \le 4$.

a. Displacement$= \int_1^4 v(t)dt$

$$= \int_1^4 (t^2 - t - 6)dt$$

$$= \frac{1}{3}t^3 - \frac{1}{2}t^2 - 6t\Big|_{t=1}^{t=4}$$

$$= \left(\frac{1}{3}(4)^3 - \frac{1}{2}(4)^2 - 6(4)\right) - (\frac{1}{3}(1)^3 - \frac{1}{2}(1)^2 - 6(1))$$

$$= -\frac{9}{2}m \text{ (to the left).}$$

b. Distance Traveled$= \int_1^4 |v(t)|dt = \int_1^4 |(t^2 - t - 6)|dt$

To integrate the absolute value of a function we need to know where the function is positive and where it's negative. We then use the fact that:

$$|f(t)| = f(t) \quad \text{if } f(t) \geq 0$$
$$= -f(t) \quad \text{if } f(t) \leq 0.$$

$$t^2 - t - 6 = (t-3)(t+2) = 0 \quad \Longrightarrow \quad t = 3, -2.$$

By testing the sign of this function on the intervals:

$t < -2, \quad -2 < t < 3, \quad 3 < t,$ we get:

sign of $t^2 - t - 6$ ____+____|______−______|____+____

-2 $\qquad$ 3

$t^2 - t - 6 \geq 0$ when $t \leq -2$ or $t \geq 3$

$t^2 - t - 6 \leq 0$ when $-2 \leq t \leq 3$.

So when $1 \leq t \leq 4$ we have:

$t^2 - t - 6 \geq 0$ when $3 \leq t \leq 4$

$t^2 - t - 6 \leq 0$ when $1 \leq t \leq 3$.

So $|(t^2 - t - 6)| = t^2 - t - 6$ when $3 \leq t \leq 4$

$= -(t^2 - t - 6)$ when $1 \leq t \leq 3$

Distance Traveled$= \int_1^4 |(t^2 - t - 6)|dt$

$$= -\int_1^3 (t^2 - t - 6)dt + \int_3^4 (t^2 - t - 6)dt$$
$$= -\left(\frac{1}{3}t^3 - \frac{1}{2}t^2 - 6t\right)\Big|_{t=1}^{t=3} + \left(\frac{1}{3}t^3 - \frac{1}{2}t^2 - 6t\right)\Big|_{t=3}^{t=4}$$
$$= \frac{61}{6}m.$$

Future Value of the Position Function

Since $\int_0^t v(x)dx = \int_0^t s'(x)dx = s(t) - s(0)$

$$\boldsymbol{s(t) = s(0) + \int_0^t v(x)dx}$$

Position at time t=initial position+displacement over $(0, t)$.

Ex. A block hangs at rest from a massless spring at the origin $(s = 0)$. At $t = 0$, the block is pulled downward $\frac{1}{2}$ meter to its initial position $s(0) = -\frac{1}{2}$ and released. Its velocity (in m/s) is given by $v(t) = \frac{1}{2}sint$, for $t \geq 0$. Assume that the upward direction is positive.

a. Find the position of the block for $t \geq 0$.
b. When does the block move through the origin for the first time?

a. $s(t) = s(0) + \int_0^t v(x)dx = -\frac{1}{2} + \int_0^t \frac{1}{2} sinxdx$

$$= -\frac{1}{2} + \frac{1}{2}(-cosx)\Big|_{x=0}^{x=t}$$

$$= -\frac{1}{2} + \frac{1}{2}(-cost + 1) = -\frac{1}{2}cost$$

b. $-\frac{1}{2}cost = 0$, for $t \geq 0$, $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots, \frac{(2n+1)\pi}{2}, \ldots$

so $t = \frac{\pi}{2}$ seconds is the first time that the block moves through the origin.

Just as we can get the position from the velocity, we can get velocity from acceleration:

Since $a(t) = v'(t)$, where $a(t)$ =acceleration

Change in velocity$= v(b) - v(a) = \int_a^b v'(t)dt = \int_a^b a(t)dt.$

If we know the initial velocity $v(0)$ we can find the future velocity for $t \geq 0$ by

$$\boldsymbol{v(t) = v(0) + \int_0^t a(x)dx}$$

Future velocity=initial velocity + change in velocity over $(0, t)$.

Ex. The acceleration of a particle is given by $a(t) = 2t - 4, \quad t \geq 0$, with initial velocity $v(0) = 3$, and initial position $s(0) = 4$. Find

a. The velocity function
b. The distance travelled for $0 \leq t \leq 2$
c. The position function.

a. $\mathrm{v(t) = v(0) + \int_0^t a(x)dx} = 3 + \int_0^t (2x - 4)dx$

$$= 3 + (x^2 - 4x)\Big|_{x=0}^{x=t}$$
$$= 3 + (t^2 - 4t) - (0 - 0)$$
$$= t^2 - 4t + 3$$

b. Distance travelled$= \int_0^2 |v(t)|dt = \int_0^2 |t^2 - 4t + 3|dt$

$$t^2 - 4t + 3 = (t - 3)(t - 1) = 0 \implies \quad t = 3, 1$$

By testing the sign of this function on the intervals:
$t < 1, \quad 1 < t < 3, \quad 3 < t,$ we get:

sign of $t^2 - 4t + 3$ ____+____|_____−_____|____+____

1 3

$t^2 - 4t + 3 \geq 0$ when $t \leq 1$ or $t \geq 3$
$t^2 - 4t + 3 \leq 0$ when $1 \leq t \leq 3$.

So on the interval $[0,2]$:

So $|t^2 - 4t + 3| = t^2 - 4t + 3$ if $0 \leq t \leq 1$

$= -(t^2 - 4t + 3)$ if $1 \leq t \leq 2.$

Distance travelled $= \int_0^1 (t^2 - 4t + 3)dt - \int_1^2 (t^2 - 4t + 3)dt$

$$= (\tfrac{1}{3}t^3 - 2t^2 + 3t)\Big|_{t=0}^{t=1} - (\tfrac{1}{3}t^3 - 2t^2 + 3t)\Big|_{t=1}^{t=2}$$

$$= \left[\left(\tfrac{1}{3} - 2 + 3\right) - (0)\right] - [\left(\tfrac{8}{3} - 8 + 6\right) - \left(\tfrac{1}{3} - 2 + 3\right)]$$

$$= \left(\tfrac{4}{3}\right) - \left(\tfrac{2}{3} - \tfrac{4}{3}\right) = 2.$$

c. $s(t) = s(0) + \int_0^t v(x)dx = 4 + \int_0^t (x^2 - 4x + 3)dx$

$$= 4 + (\tfrac{1}{3}x^3 - 2x^2 + 3x)\Big|_{x=0}^{x=t}$$

$$= 4 + \left(\tfrac{1}{3}t^3 - 2t^2 + 3t\right) - 0$$

$$= \tfrac{1}{3}t^3 - 2t^2 + 3t + 4.$$

The relationships that hold for velocity, position and displacement also hold for more general situations. That is, if $Q(t)$ is the amount of something (e.g. a population) and $Q'(t)$ is the rate at which it is changing then:

$$\int_a^b \boldsymbol{Q'(t)dt = Q(b) - Q(a) = Net\ Change\ in\ Q\ over}\ [\boldsymbol{a}, \boldsymbol{b}].$$

We can also get the future value of $Q(t)$ by:

$$\boldsymbol{Q(t) = Q(0) + \int_0^t Q'(x)dx}.$$

Ex. The population of a rural town was 250 people in 2010. During the following years, the population grew at a rate of $P'(t) = 30(1 + \sqrt{t})$, where t is in years.

a. Approximate the population in 2030?
b. Find the population $P(t)$ at any time after 2010.

a. $\int_0^{20} 30\left(1 + \sqrt{t}\right)dt = net\ change\ in\ population\ from\ 2010\ to\ 2030$

$$= 30t + 20t^{\frac{3}{2}}\Big|_{t=0}^{t=20}$$

$$= 600 + 20\left(20\sqrt{20}\right) = 600 + 400\sqrt{20}$$

$$\approx 2{,}389$$

So the total population in 2030 would be approximately $2{,}389 + 250 = 2{,}639$.

b. $\mathrm{P(t) = P(0) + \int_0^t P'(x)dx}$

$$= 250 + \int_0^t 30(1 + \sqrt{x})dx$$

$$= 250 + (30x + 20x^{\frac{3}{2}})\Big|_{x=0}^{x=t}$$

$$= 250 + 30t + 20t^{\frac{3}{2}}.$$

Ex. Starting with an initial population of $P(0) = 50$, a population of cats grows at a rate of $P'(t) = 25 - \frac{t}{4}$ (in cats per month), for $0 \le t \le 100$.

a. What is the population after 8 months?
b. What is the population $P(t)$ for $0 \le t \le 100$?

a. $$\begin{aligned} \mathrm{P}(8) &= \mathrm{P}(0) + \int_0^8 \mathrm{P}'(\mathrm{t})\mathrm{dt} \\ &= 50 + \int_0^8 \left(25 - \frac{t}{4}\right) dt \\ &= 50 + \left(25t - \frac{1}{8}t^2\right)\Big|_{t=0}^{t=8} \\ &= 50 + (200 - 8) = 242. \end{aligned}$$

c. b. $$\begin{aligned} \mathrm{P}(\mathrm{t}) &= \mathrm{P}(0) + \int_0^{\mathrm{t}} \mathrm{P}'(\mathrm{x})\mathrm{dx} \\ &= 50 + \int_0^t \left(25 - \frac{x}{4}\right) dx \\ &= 50 + \left(25x - \frac{1}{8}x^2\right)\Big|_{x=0}^{x=t} \\ &= 50 + 25t - \frac{1}{8}t^2. \end{aligned}$$

Ex. An oil refinery produces oil at a rate given by:

$$Q'(t) = \begin{cases} 800 & 0 \le t < 30 \\ 2600 - 60t & 30 \le t \le 40 \\ 200 & 40 < t \end{cases}$$

where t is in days and $Q(t)$ is in barrels.

a. How many barrels of oil are produced in the first 35 days?

b. How many barrels of oil are produced in the first 50 days?

a. $Q(t) = Q(0) + \int_0^t Q'(x)dx$. Over the first 35 days notice that:

$$Q'(t) = \begin{cases} 800 & 0 \le t < 30 \\ 2600 - 60t & 30 \le t \le 35. \end{cases}$$

$$Q(35) = Q(0) + \int_0^{35} Q'(x)dx$$

$$= 0 + \int_0^{30} 800dx + \int_{30}^{35}(2600 - 60x)dx$$

$$= 800x|_{x=0}^{x=30} + (2600x - 30x^2)|_{x=30}^{x=35}$$

$$= 24000 + [(2600(35) - 30(35)^2) - (2600(30) - 30(30)^2]$$

$= 27{,}250$ barrels.

b. $Q(t) = Q(0) + \int_0^t Q'(x)dx$. Over the first 50 days notice that:

$$Q'(t) = \begin{cases} 800 & 0 \le t < 30 \\ 2600 - 60t & 30 \le t \le 40 \\ 200 & 40 < t \le 50. \end{cases}$$

$$Q(50) = Q(0) + \int_0^{50} Q'(x)dx$$

$$= 0 + \int_0^{30} 800dx + \int_{30}^{40}(2600 - 60x)dx + \int_{40}^{50} 200dx$$

$$= 800x|_{x=0}^{x=30} + (2600x - 30x^2)|_{x=30}^{x=40} + 200x|_{x=40}^{x=50}$$

$$= 24000 + [(2600(40) - 30(40)^2) - (2600(30) - 30(30)^2]$$
$$+(200(50) - 200(40))$$

$= 24000 + 5000 + 2000 = 31,000$ barrels.