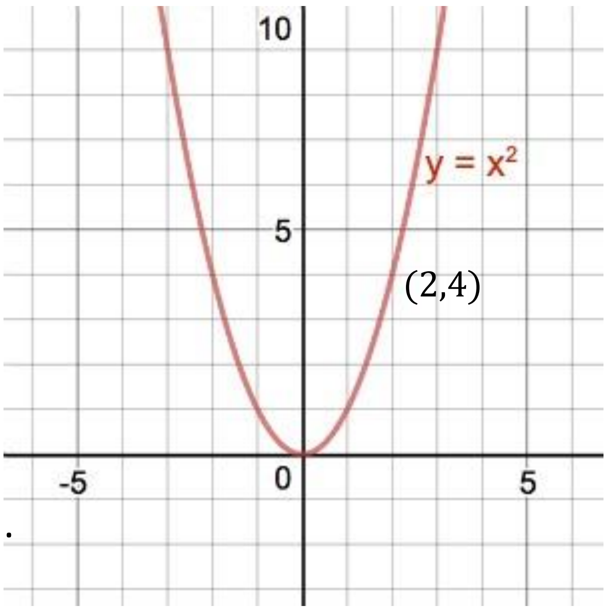
Limits and One-sided Limits

Informal definition: $\lim_{x \to a} f(x) = L$; The limit as x goes to a of $f(x)$ is L means as x tends toward a, $f(x)$ tends toward L.

Ex. $f(x) = x^2$; what is $\lim_{x \to 2} x^2$?



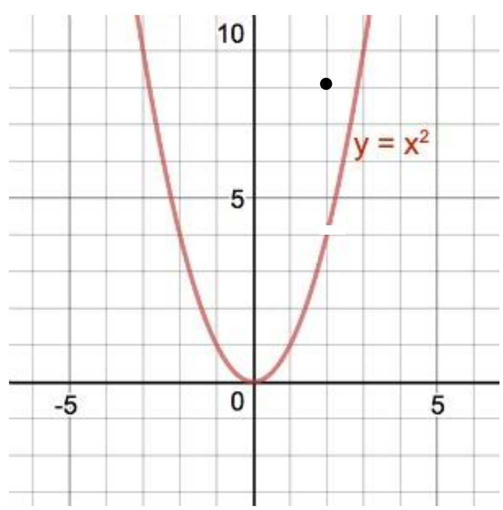
x	$f(x)$
1	1
1.5	2.25
1.9	3.61
1.99	3.9601
1.999	3.9960

x	$f(x)$
3	9
2.5	6.25
2.1	4.41
2.01	4.0401
2.001	4.0040

Notice that we don't care what the value of the function is at $x = a$, we only care what the value of the function is tending toward as x approaches a.

Ex. $f(x) = x^2 \quad if \;\; x \neq 2$

$\quad\quad = 8 \quad if \;\; x = 2$



Even though $f(2) = 8$, $\lim_{x \to 2} f(x) = 4$.

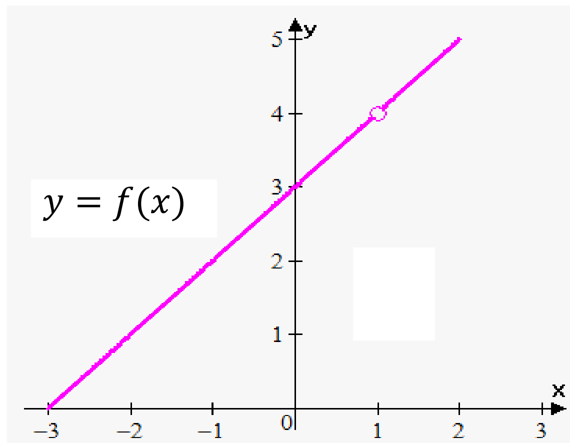
Ex. What is $\lim\limits_{x\to 1} \frac{x^2+2x-3}{x-1}$?

Notice that $f(x) = \frac{x^2+2x-3}{x-1}$ is only defined for $x \neq 1$ (since the denominator can't be 0). So this function isn't even defined at the point where we want to take the limit (i.e., $x = 1$).

Notice also that when $x \neq 1$, $f(x) = \frac{x^2+2x-3}{x-1} = \frac{(x-1)(x+3)}{(x-1)} = x + 3$, so we might expect that $\lim\limits_{x\to 1} \frac{x^2+2x-3}{x-1} = \lim\limits_{x\to 1}(x+3) = 4$.

x	$f(x)$
.5	3.5
.9	3.9
.99	3.99
.999	3.999
.9999	3.999

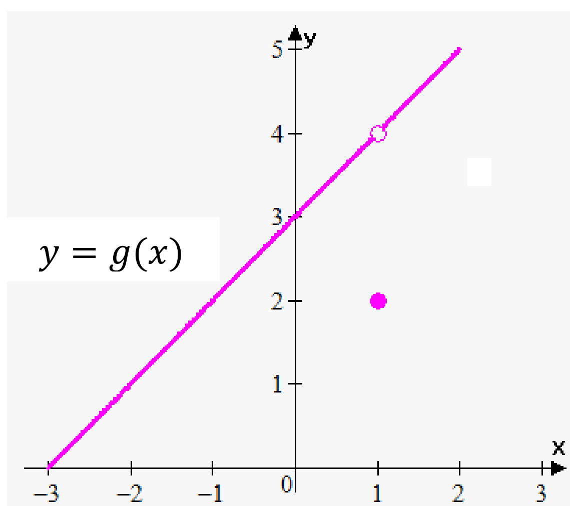
x	$f(x)$
1.5	4.5
1.1	4.1
1.01	4.01
1.001	4.001
1.0001	4.0001



Ex. Suppose $g(x) = \frac{x^2+2x-3}{x-1}$ $\quad x \neq 1$

$\qquad\qquad = 2 \qquad x = 1$

What is $\lim\limits_{x\to 1} g(x)$?



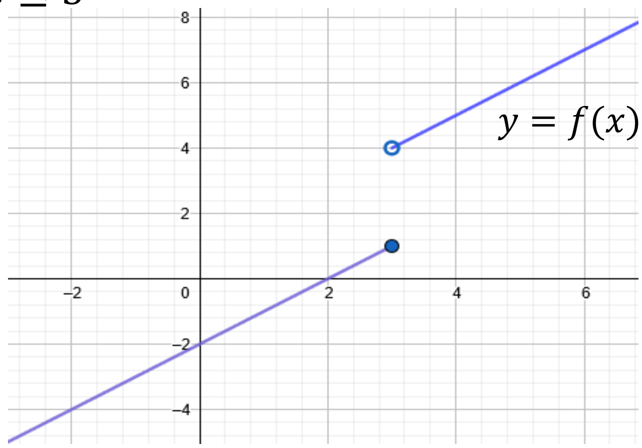
$\lim\limits_{x\to 1} g(x) = 4$; but $g(1) = 2$.

Ways in which a function can fail to have a limit at $x = a$:

1. The function approaches different values from the right and the left

   Ex. $f(x) = x + 1 \qquad x > 3$
   $\qquad\quad = x - 2 \qquad x \leq 3$

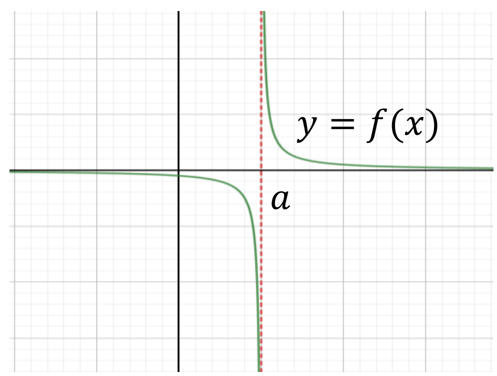
   $\lim_{x\to 2} f(x)$ does not exist.



2. The function tends toward positive or negative infinity at $x = a$.

   Ex. $f(x) = \frac{1}{x-a} \qquad x \neq a$

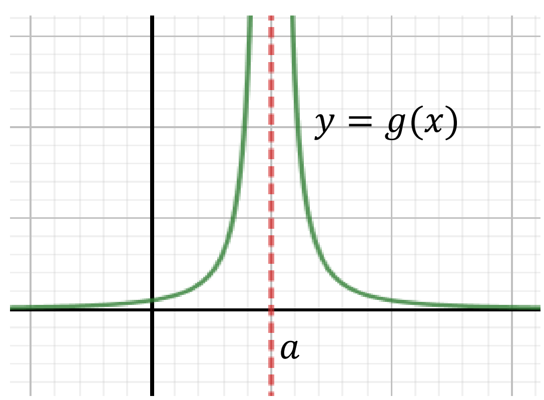
   $\lim_{x\to a} f(x)$ does not exist.



   Ex. $g(x) = \frac{1}{(x-a)^2} \qquad x \neq a$

   $\lim_{x\to a} g(x)$ does not exist.

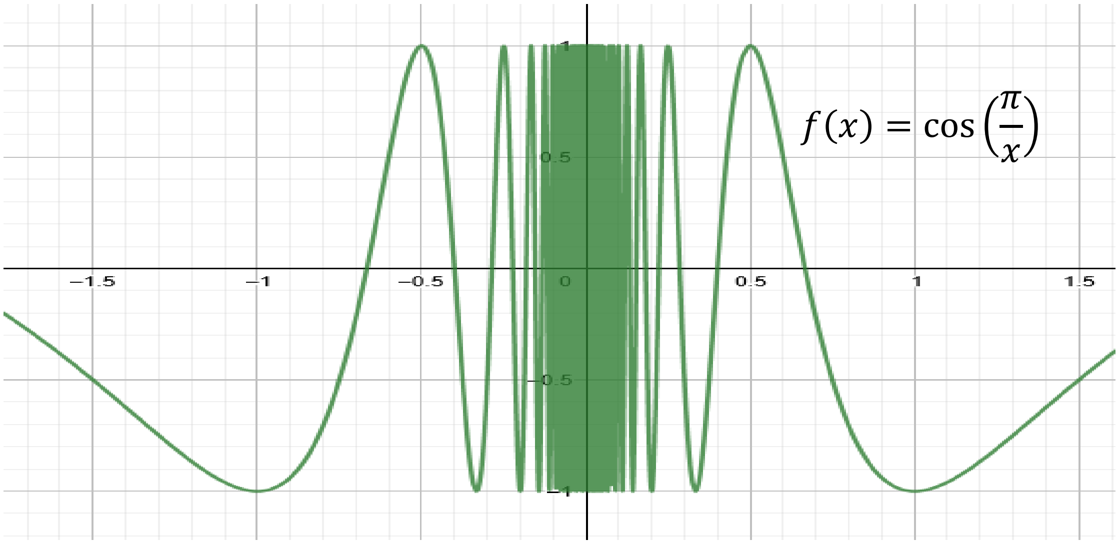
   Later we will talk about what it means to say $\lim_{x\to a} g(x) = \infty$ or $-\infty$.



3. The function "wiggles" up and down "too much"

   Ex. $f(x) = \cos\left(\frac{\pi}{x}\right) \qquad x \neq 0$

   $\lim_{x \to 0} f(x)$ does not exist.



<u>One-sided Limits</u>

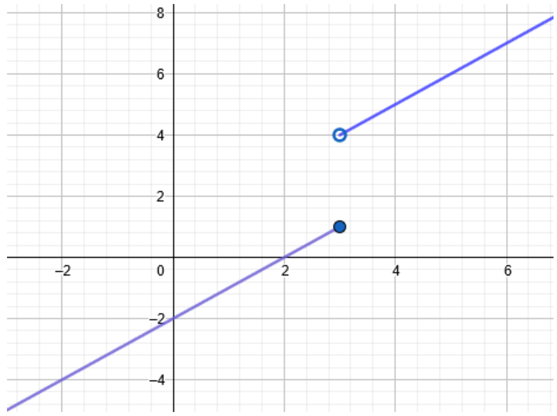
Informal definition: We say $\lim_{x \to a^-} f(x) = L$, **The limit of $f(x)$ as x approaches a from the left is equal to L**, if the values of $f(x)$ tend to L as x approaches a from the left (ie from lower values of x).

We say $\lim_{x \to a^+} f(x) = L$, **The limit of $f(x)$ as x approaches a from the right is equal to L**, if the values of $f(x)$ tend to L as x approaches a from the right (ie from higher values of x).

For $\lim_{x \to a} f(x)$ to exist we must have $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

Ex. Suppose $f(x) = x + 1 \quad\quad x > 3$

$\quad\quad\quad\quad\quad\;\; = x - 2 \quad\quad x \leq 3.$

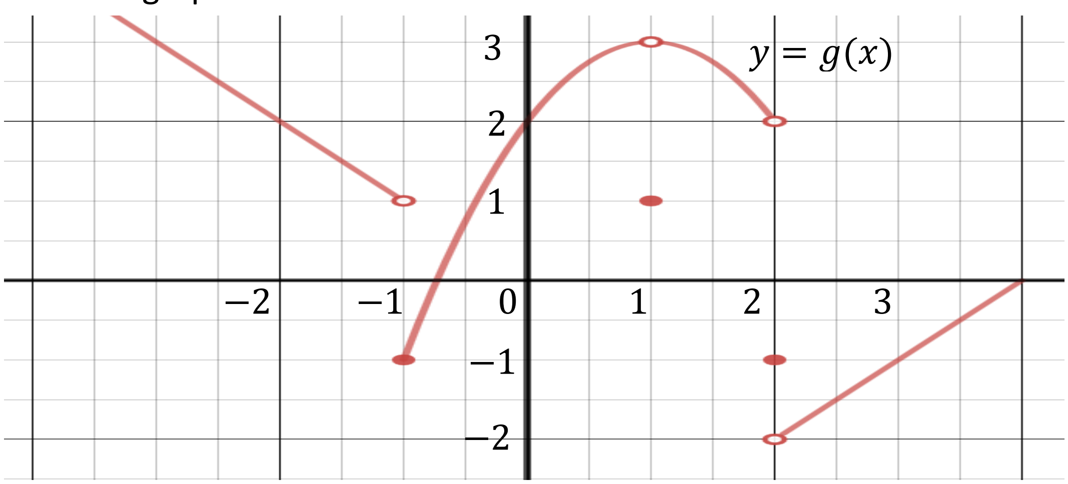


We just noted that $\lim_{x \to 2} f(x)$ does not exist, however

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x - 2) = 1$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x + 1) = 4.$

Ex. For the graph below notice that:



$\lim_{x \to -1^+} g(x) = -1$	$\lim_{x \to 1^+} g(x) = 3$	$\lim_{x \to 2^+} g(x) = -2$
$\lim_{x \to -1^-} g(x) = 1$	$\lim_{x \to 1^-} g(x) = 3$	$\lim_{x \to 2^-} g(x) = 2$
$\lim_{x \to -1} g(x) = DNE$	$\lim_{x \to 1} g(x) = 3$	$\lim_{x \to 2} g(x) = DNE$
$g(-1) = -1$	$g(1) = 1$	$g(2) = -1$.