The Substitution Rule

Not every differentiable function has an elementary antiderivative. For example, there is no elementary function for $\int \sin(x^2)\,dx$. We know a handful of basic antiderivatives (e.g. $\int x^n dx,\;\; n \neq -1,\;\;\; \int (sinax)dx$, etc.). So far we have tried to turn all of our indefinite integral problems into sums, differences, and constant multiples of these basic antiderivatives. In this section we will learn to make substitutions in indefinite integrals that will turn some integrands into sums, differences, and constant multiples of the basic antiderivatives we know.

Suppose we have a composite function $F(g(x))$, where $F(x)$ is an antiderivative of $f(x)$, i.e. $F'(x) = f(x)$ and $\int f(x)dx = F(x) + C$. [For example, $F(x) = x^4$, $g(x) = x^2 + 1$, and $f(x) = 4x^3$. So $F\big(g(x)\big) = (x^2+1)^4$.] **By the chain rule**:

$$\frac{d}{dx}F\big(g(x)\big) = F'\big(g(x)\big)g'(x) = f\big(g(x)\big)g'(x).$$

Thus we have $\int f\big(g(x)\big)g'(x)dx = F\big(g(x)\big) + C.$

In other words, faced with an integral that looks like $\int f\big(g(x)\big)g'(x)dx$ (for example $\int 4(x^2+1)^3(2x)dx$) we can make a substitution

$$u = g(x)$$

$$du = g'(x)dx$$

and $\int f\big(g(x)\big)g'(x)dx$ becomes $\int f(u)du = F(u) + C = F\big(g(x)\big) + C.$

Ex. Evaluate $\int 4(x^2+1)^3(2x)dx$.

Let $u = x^2 + 1$

$du = 2xdx$

So the integral becomes:

$$\int 4(x^2+1)^3(2x)dx = \int 4u^3du$$
$$= u^4 + C$$
$$= (x^2+1)^4 + C$$

Notice $\frac{d}{dx}((x^2+1)^4 + C) = 4(x^2+1)^3(2x)$.

Ex. Evaluate $\int 3x^2(x^3+4)^6\,dx$.

Let $u = x^3 + 4$

$du = 3x^2dx$

Now substituting into the integral:

$$\int u^6\,du = \frac{1}{7}u^7 + C$$
$$= \frac{1}{7}(x^3+4)^7 + C.$$

Ex. Evaluate $\int 3x^2 \sin(x^3)\, dx$.

Let $u = x^3$

$du = 3x^2 dx$

$$\int 3x^2 \sin(x^3)\, dx = \int (sinu)\, du$$

$$= -cosu + C$$

$$= -\cos(x^3) + C.$$

Ex. Evaluate $\int x^2(x^3+3)^{10} dx$.

Notice that x^2 is not quite the derivative of $x^3 + 3$, but it is up to a constant multple. That's close enough.

Let $u = x^3 + 3$

$du = 3x^2 dx$

$\frac{1}{3} du = x^2 dx$

$$\int x^2(x^3+3)^{10} dx = \int u^{10}\left(\frac{1}{3}\right) du$$

$$= \frac{1}{3}\int u^{10} du$$

$$= \frac{1}{3}\left(\frac{1}{11}\right) u^{11} + C$$

$$= \frac{1}{33}(x^3+3)^{11} + C$$

Ex. Evaluate $\int \frac{1}{\sqrt{5x+2}}dx$.

$$\int \frac{1}{\sqrt{5x+2}}dx = \int (5x+2)^{-\frac{1}{2}}dx$$

Let $u = 5x + 2$

$$du = 5dx$$

$$\frac{1}{5}du = dx$$

$$\int (5x+2)^{-\frac{1}{2}}dx = \int u^{-\frac{1}{2}}\left(\frac{1}{5}\right)du$$

$$= \frac{1}{5}\int u^{-\frac{1}{2}}du$$

$$= \frac{1}{5}\left(2u^{\frac{1}{2}}\right) + C$$

$$= \frac{2}{5}(5x+2)^{\frac{1}{2}} + C$$

Ex. Evaluate $\int \frac{3x}{\sqrt{1-x^2}}dx$.

$$\int \frac{3x}{\sqrt{1-x^2}}dx = \int 3x(1-x^2)^{-\frac{1}{2}}dx$$

Let $u = 1 - x^2$

$$du = -2xdx$$

$$-\frac{1}{2}du = xdx$$

$$\int 3x(1-x^2)^{-\frac{1}{2}}\,dx = \int 3u^{-\frac{1}{2}}\left(-\frac{1}{2}\right)du$$

$$= -\frac{3}{2}\int u^{-\frac{1}{2}}du$$

$$= -\left(\frac{3}{2}\right)\left(2u^{\frac{1}{2}}\right) + C$$

$$= -3(1-x^2)^{\frac{1}{2}} + C$$

Ex. Evaluate $\int sin2x(cos^{10}2x)dx$

Let $u = cos2x$

$$du = (-2sin2x)dx$$

$$-\frac{1}{2}du = (sin2x)dx$$

$$\int sin2x(cos^{10}2x)dx = \int u^{10}\left(-\frac{1}{2}\right)du$$

$$= -\frac{1}{2}\int u^{10}du$$

$$= -\frac{1}{2}\left(\frac{1}{11}\right)u^{11} + C$$

$$= -\frac{1}{22}(cos2x)^{11} + C.$$

$$= -\frac{1}{22}\cos^{11}2x + C.$$

Ex. Evaluate $\int x\sqrt{x+7}dx$

Let $u = x + 7$ and thus $x = u - 7.$

$du = dx$

$\int x\sqrt{x+7}dx = \int (u-7)u^{\frac{1}{2}}du$ now multiply

$= \int (u^{\frac{3}{2}} - 7u^{\frac{1}{2}})du$

$= \frac{2}{5}u^{\frac{5}{2}} - 7\left(\frac{2}{3}u^{\frac{3}{2}}\right) + C$

$= \frac{2}{5}(x+7)^{\frac{5}{2}} - \frac{14}{3}(x+7)^{\frac{3}{2}} + C$

Ex. Evaluate $\int \frac{2x}{\sqrt[3]{x-4}}dx$

$\int \frac{2x}{\sqrt[3]{x-4}}dx = \int 2x(x-4)^{-\frac{1}{3}}dx$

let $u = x - 4 \implies x = u + 4$

$du = dx$

$\int 2x(x-4)^{-\frac{1}{3}}dx = \int 2(u+4)u^{-\frac{1}{3}}du$ now multiply

$= 2\int \left(u^{\frac{2}{3}} + 4u^{-\frac{1}{3}}\right)du$

$= 2\left(\frac{3}{5}u^{\frac{5}{3}} + 4\left(\frac{3}{2}\right)u^{\frac{2}{3}}\right) + C$

$= \frac{6}{5}(x-4)^{\frac{5}{3}} + 12(x-4)^{\frac{2}{3}} + C.$

Definite Integrals: Changing the endpoints

Ex. Evaluate $\int_{x=0}^{x=2} x\sqrt{x^2+5}dx$

Approach #1: Let $u = x^2 + 5$; *when* $x = 0$, *then* $u = 0^2 + 5 = 5$,

$du = 2xdx$ when $x = 2$, then $u = 2^2 + 5 = 9$

$\frac{1}{2}du = dx$

$$\int_{x=0}^{x=2} x\sqrt{x^2+5}dx = \int_{u=5}^{u=9}(u)^{\frac{1}{2}}\left(\frac{1}{2}\right)du = \frac{1}{2}\left(\frac{2}{3}\right)u^{\frac{3}{2}}\Big|_{u=5}^{u=9} = \frac{1}{3}(9^{\frac{3}{2}} - 5^{\frac{3}{2}})$$

$$= \frac{1}{3}(27 - 5^{\frac{3}{2}})$$

OR Approach #2

Evaluate the indefinite integral completely and then substitute the original endpoints of integration.

$\int_{x=0}^{x=2} x\sqrt{x^2+5}dx$; Evaluate $\int x\sqrt{x^2+5}dx$ first.

Let $u = x^2 + 5$

$du = 2xdx$

$\frac{1}{2}du = dx$

$$\int x\sqrt{x^2+5}dx = \int u^{\frac{1}{2}}\left(\frac{1}{2}\right)du = \frac{1}{2}\left(\frac{2}{3}\right)u^{\frac{3}{2}}$$

$$= \frac{1}{3}u^{\frac{3}{2}} + C$$

$$= \frac{1}{3}(x^2+5)^{\frac{3}{2}} + C$$

So $\int_{x=0}^{x=2} x\sqrt{x^2+5}dx = \frac{1}{3}(x^2+5)^{\frac{3}{2}}\Big|_{x=0}^{x=2} = \frac{1}{3}\left(9^{\frac{3}{2}} - 5^{\frac{3}{2}}\right) = \frac{1}{3}(27 - 5^{\frac{3}{2}})$.

Ex. Evaluate $\int_1^2 x(x^2-2)^3 dx$

#1: Let $u = x^2 - 2$; when $x = 1$, then $u = 1^2 - 2 = -1$

$du = 2xdx$; when $x = 2$, then $u = 2^2 - 2 = 2$

$\frac{1}{2} du = xdx$

$$\int_{x=1}^{x=2} x(x^2-2)^3 dx = \int_{u=-1}^{u=2} u^3 \left(\frac{1}{2}\right) du$$

$$= \frac{1}{2}\int_{u=-1}^{u=2} u^3 \, du$$

$$= \frac{1}{2}\left(\frac{1}{4}\right) u^4 \Big|_{u=-1}^{u=2}$$

$$= \frac{1}{8}(2^4 - (-1)^4) = \frac{15}{8}$$

#2: Evaluate $\int x(x^2-2)^3 dx$ first.

Let $u = x^2 - 2$

$du = 2xdx$

$\frac{1}{2} du = xdx$

$$\int x(x^2-2)^3 dx = \int u^3 \left(\frac{1}{2}\right) du$$

$$= \frac{1}{2}\int u^3 du$$

$$= \frac{1}{2}\left(\frac{1}{4}u^4\right) + C = \frac{1}{8}(x^2-2)^4 + C$$

$$\int_1^2 x(x^2-2)^3 dx = \frac{1}{8}(x^2-2)^4 \Big|_{x=1}^{x=2}$$

$$= \frac{1}{8}[(2^2-2)^4 - (1^2-2)^4] = \frac{15}{8}.$$

Ex. Evaluate $\int_0^{\frac{\pi}{4}} \frac{sin\theta}{cos^3\theta} d\theta$

#1: $\int_{\theta=0}^{\theta=\frac{\pi}{4}} \frac{sin\theta}{cos^3\theta} d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{4}} (sin\theta)(cos\theta)^{-3} d\theta$

Let $u = cos\theta$; when $\theta = 0,\ cos\theta = 1$

$du = -sin\theta d\theta$ when $\theta = \frac{\pi}{4},\ cos\theta = \frac{\sqrt{2}}{2}$

$-du = sin\theta d\theta$

$$\int_{\theta=0}^{\theta=\frac{\pi}{4}} (sin\theta)(cos\theta)^{-3} d\theta = \int_{u=1}^{u=\frac{\sqrt{2}}{2}} u^{-3}(-1)du = \frac{1}{2}u^{-2}\Big|_{u=1}^{u=\frac{\sqrt{2}}{2}}$$

$$= \frac{1}{2}\left(\frac{1}{(\frac{\sqrt{2}}{2})^2} - \frac{1}{1^2}\right) = \frac{1}{2}(2-1) = \frac{1}{2}.$$

#2: Evaluate $\int \frac{sin\theta}{cos^3\theta} d\theta$ first.

$$\int \frac{sin\theta}{cos^3\theta} d\theta = \int (sin\theta)(cos\theta)^{-3} d\theta$$

Let $u = cos\theta$;

$du = -sin\theta d\theta$

$-du = sin\theta d\theta$

$$\int \frac{sin\theta}{cos^3\theta} d\theta = \int (sin\theta)(cos\theta)^{-3} d\theta = \int u^{-3}(-1)du$$

$$= \frac{1}{2}u^{-2} + C$$

$$= \frac{1}{2}(cos\theta)^{-2} + C$$

So $\int_0^{\frac{\pi}{4}} \frac{sin\theta}{cos^3\theta} d\theta = \frac{1}{2}(cos\theta)^{-2}\Big|_{\theta=0}^{\theta=\frac{\pi}{4}}$

$$= \frac{1}{2}\left(\frac{1}{(\cos(\frac{\pi}{4}))^2} - \frac{1}{(\cos(0))^2}\right) = \frac{1}{2}(2-1) = \frac{1}{2}.$$

<u>$\int sin^2x\,dx$ and $\int cos^2x\,dx$</u>

We need the following identities for these integrals

$$sin^2x = \frac{1-cos2x}{2} \quad , \qquad cos^2x = \frac{1+cos2x}{2}$$

$$\int sin^2x\,dx = \int \frac{1-cos2x}{2}dx = \int \frac{1}{2}dx - \frac{1}{2}\int cos2xdx$$

$$= \frac{1}{2}x - \frac{1}{2}\left(\frac{1}{2}sin2x\right)$$

$$= \frac{1}{2}x - \frac{1}{4}sin2x + C.$$

$$\int cos^2x\,dx = \int \frac{1+cos2x}{2}dx = \int \frac{1}{2}dx + \frac{1}{2}\int cos2xdx$$

$$= \frac{1}{2}x + \frac{1}{2}\left(\frac{1}{2}sin2x\right) + C$$

$$= \frac{1}{2}x + \frac{1}{4}sin2x + C.$$

Ex. Evaluate $\int_0^{\frac{\pi}{2}} 8sin^2x\,dx$.

$$\int_0^{\frac{\pi}{2}} 8sin^2x\,dx = 8\int_0^{\frac{\pi}{2}}\left(\frac{1}{2}-\frac{1}{2}cos2x\right)dx$$

$$= 8(\frac{1}{2}x-\frac{1}{4}sin2x)\Big|_{x=0}^{x=\frac{\pi}{2}}$$

$$= 8\left[\left(\frac{\pi}{4}-\frac{1}{4}sin\pi\right)-(0)\right] = 2\pi.$$

Ex. Evaluate $\int_0^{\frac{\pi}{6}} 24cos^2(x)dx$.

$$\int_0^{\frac{\pi}{6}} 24cos^2(x)dx = 24\int_0^{\frac{\pi}{6}}\left(\frac{1}{2}+\frac{1}{2}cos2x\right)dx$$

$$= 24(\frac{1}{2}x+\frac{1}{4}sin2x)\Big|_{x=0}^{x=\frac{\pi}{6}}$$

$$= 24[\left(\frac{1}{2}\left(\frac{\pi}{6}\right)+\frac{1}{4}\sin\left(2\left(\frac{\pi}{6}\right)\right)\right)-\left(\frac{1}{2}(0)+\frac{1}{4}sin0\right)]$$

$$= 24[(\frac{\pi}{12}+\frac{1}{4}\left(\sin\left(\frac{\pi}{3}\right)\right)-0]$$

$$= 24(\frac{\pi}{12}+\frac{1}{4}\left(\frac{\sqrt{3}}{2}\right))$$

$$= 2\pi + 3\sqrt{3}.$$