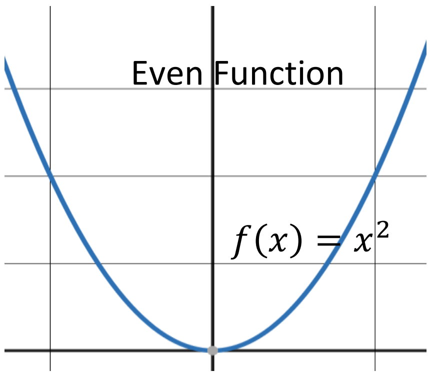
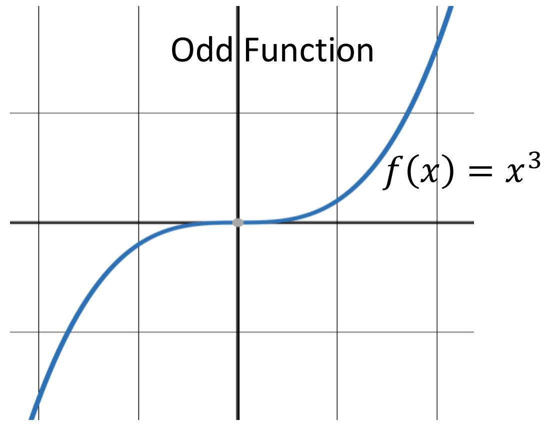
More Properties of Integrals

Integrating Even and Odd Functions:

Def. A function is even if $f(-x) = f(x)$. A function is odd if $f(-x) = -f(x)$.





Ex. Determine whether each of the functions is even, odd, or neither.

a. $f(x) = x^5 + 2x^3$
b. $f(x) = x^5 + 2x^3 + 1$
c. $f(x) = \frac{1}{x^4+2x^2+1}$
d. $f(x) = sin^3 x$
e. $f(x) = cos^3 x$

a. $f(-x) = (-x)^5 + 2(-x)^3 = -x^5 - 2x^3 = -f(x);$ odd.

b. $f(-x) = (-x)^5 + 2(-x)^3 + 1 = -x^5 - 2x^3 + 1;$ neither.

c. $f(-x) = \frac{1}{(-x)^4+2(-x)^2+1} = \frac{1}{x^4+2x^2+1} = f(x);$ even.

d. $f(-x) = sin^3(-x) = (\sin(-x))^3 = (-sinx)^3$

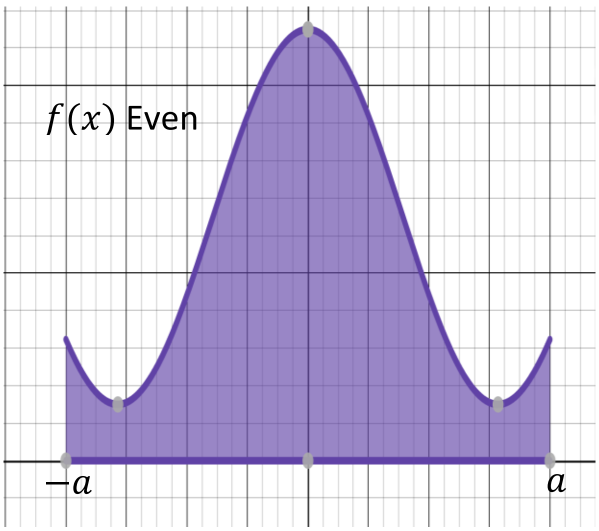
$= -\sin^3 x = -f(x);$ odd.

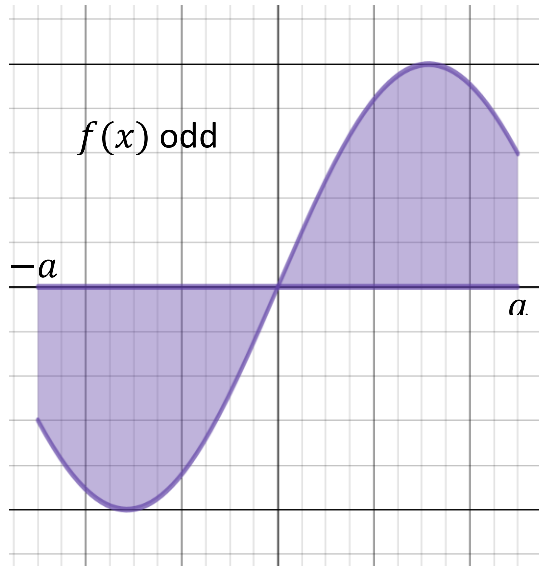
e. $f(-x) = cos^3(-x) = (\cos(-x))^3 = (cosx)^3 = f(x);$ even.

Theorem: Let a be a positive real number and let f be an integrable function on the interval $[-a, a]$.

If f is even then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

If f is odd then $\int_{-a}^{a} f(x)dx = 0$.





Ex. Evaluate the following definite integrals using symmetry (f odd/even).

a. $\int_{-2}^{2}(2x^3 - 3x^2)dx$

b. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(2sin^3x - 3cosx)dx$

c. $\int_{-3}^{3} \frac{\sin(4x)}{x^6+3} dx$

d. $\int_{-2}^{2}(3 + |x|)dx$

a. $\int_{-2}^{2}(2x^3 - 3x^2)dx = 2\int_{-2}^{2} x^3dx - 3\int_{-2}^{2} x^2dx$

x^3 is odd and x^2 is even so

$$= 0 - 3(2)\int_{0}^{2} x^2dx$$

$$= 6\left(\frac{1}{3}x^3\right)\Big|_0^2 = 2(2^3 - 0^3) = 16.$$

b. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(2sin^3x - 3cosx)dx = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sin^3xdx - 3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cosxdx$

sin^3x is odd and $cosx$ is even so

$$= 0 - 3(2)\int_0^{\frac{\pi}{2}} cosxdx$$
$$= -6(sinx)|_0^{\pi/2}$$
$$= -6\left[\left(sin\frac{\pi}{2}\right) - sin0\right]$$
$$= -6(1-0) = -6.$$

c. $\int_{-3}^{3} \frac{\sin{(4x)}}{x^6+3} dx = 0;$ since $\frac{\sin(-4x)}{(-x)^6+3} = -\frac{\sin(4x)}{x^6+3};$ $\frac{\sin(4x)}{x^6+3}$ is odd.

d. $\int_{-2}^{2}(3+|x|)dx\,;$ if $f(x) = 3 + |x|$ then

$f(-x) = 3 + |-x| = 3 + |x| = f(x)$

so $f(x)$ is even.

$$\int_{-2}^{2}(3+|x|)dx = 2\int_0^2(3+|x|)dx = 2\int_0^2(3+x)dx;$$ since $0 \leq x.$

$$= 2\left(3x + \frac{1}{2}x^2\right)\Big|_0^2$$
$$= 2[\left(3(2) + \frac{1}{2}(2)^2\right) - \left(3(0) + \frac{1}{2}(0)^2\right)]$$
$$= 2(6+2)$$
$$= 16.$$

Ex. Suppose we know that $\int_{-3}^{0} f(x)dx = 4$:

a. If $f(-x) = f(x)$ evaluate $\int_{-3}^{3} f(x)dx$.
b. If $f(-x) = -f(x)$ evaluate $\int_{-3}^{3} f(x)dx$.

a. $\int_{-3}^{3} f(x)dx = 2\int_{-3}^{0} f(x)dx = 2(4) = 8$

b. $\int_{-3}^{3} f(x)dx = 0$; since $f(x)$ is odd.

<u>Average Value of a Function</u>

The average value of n number $y_1, y_2, \ldots y_n$ is:

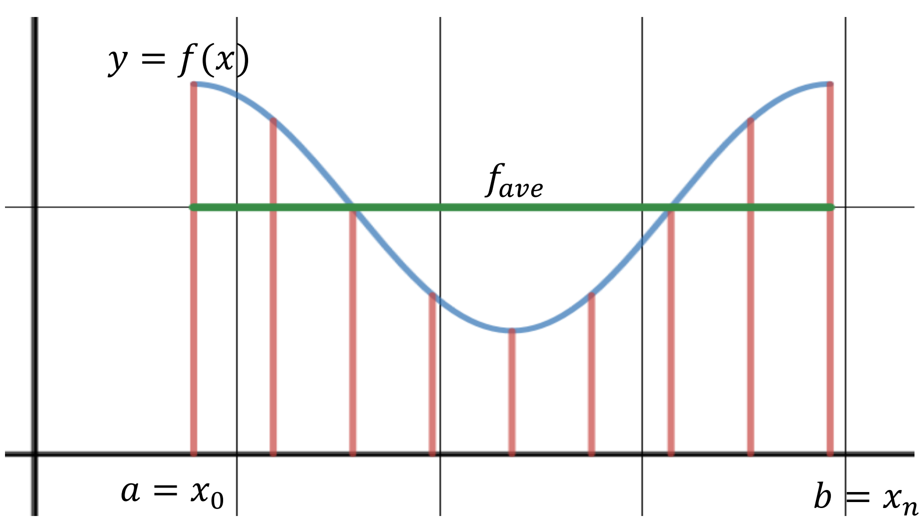
$$y_{ave} = \frac{y_1+y_2+y_3+\cdots+y_n}{n}$$

Now take an interval $[a, b]$ and divide it into n equal subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n].$$

If we take right endpoints of a function $y = f(x)$ we have:

$$Ave\ Value\ of\ f(x) \approx \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}.$$



Now multiply the top and bottom by $(b-a)$:

$$Ave\ Value\ of\ f(x) \approx \left[\frac{\big(f(x_1)+f(x_2)+\cdots+f(x_n)\big)(b-a)}{n}\right]\frac{b-a}{b-a}$$

Since $\frac{b-a}{n} = \Delta x$, we have

$$Ave\ Value\ of\ f(x) \approx \frac{1}{b-a}\big(\big(f(x_1)+f(x_2)+\cdots+f(x_n)\big)\frac{b-a}{n}$$

$$Ave\ Value\ of\ f(x) \approx \frac{1}{b-a}\big(\big(f(x_1)+f(x_2)+\cdots+f(x_n)\big)\Delta x.$$

We define the average value of the function $f(x)$ over the interval $[a,b]$ to be:

$$Ave\ Value\ of\ f(x) = \lim_{\Delta x \to 0}\frac{1}{b-a}\big(\big(f(x_1)+f(x_2)+\cdots+f(x_n)\big)\Delta x$$

$$\boldsymbol{Ave\ Value\ of\ f(x) = \bar{f} = \frac{1}{b-a}\int_a^b f(x)dx.}$$

Ex. The surface of a water wave is described by $y = 5(1+cosx)$, for $-\pi \le x \le \pi$. Find the average height of the wave on $[-\pi,\pi]$.

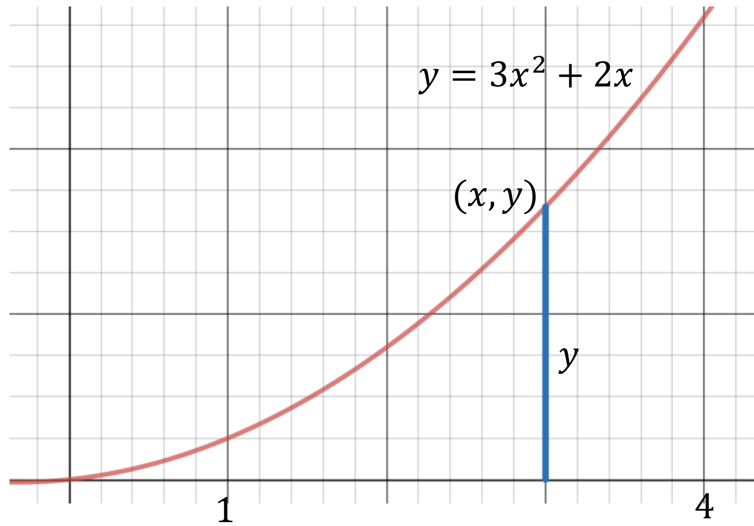
$$\begin{aligned}\bar{\mathrm{f}} = \frac{1}{\mathrm{b}-\mathrm{a}}\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{f(x)dx} &= \frac{1}{\pi-(-\pi)}\int_{-\pi}^{\pi} 5(1+cosx)dx \\ &= \frac{1}{2\pi}(5)\int_{-\pi}^{\pi}(1+cosx)dx \\ &= \frac{5}{2\pi}(x+sinx)\Big|_{-\pi}^{\pi} \\ &= \frac{5}{2\pi}[(\pi+sin\pi)-(-\pi+\sin(-\pi))] \\ &= \frac{5}{2\pi}(2\pi) = 5.\end{aligned}$$

Ex. Find the average distance of points on the parabola $y = 3x^2 + 2x$ from the x axis when $1 \le x \le 4$.

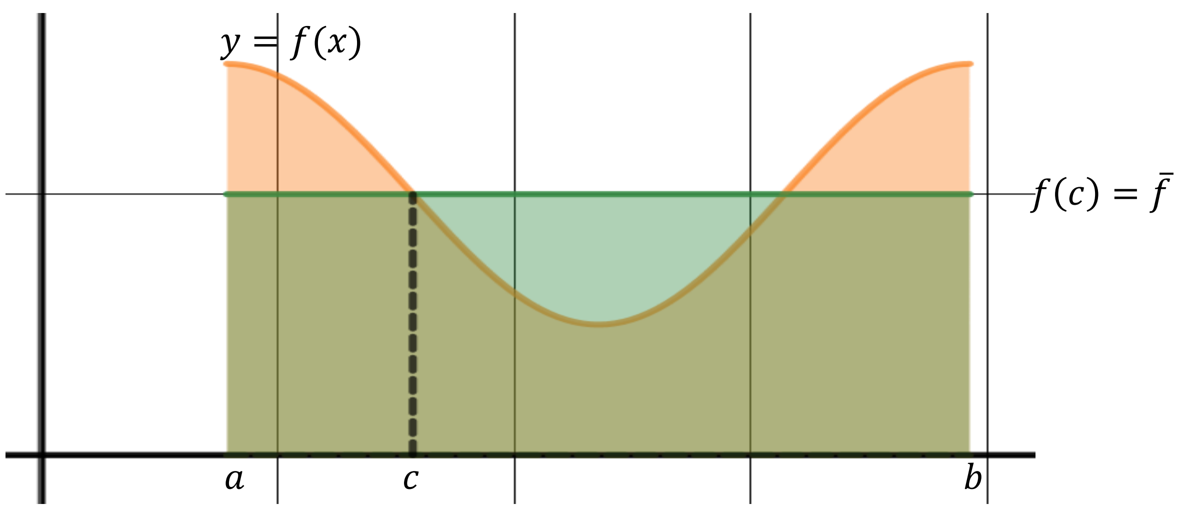
Since $1 \le x \le 4$, the y values on the parabola are all positive. Thus the distance to the x axis is just the y coordinate.

Average Distance$= \frac{1}{4-1}\int_1^4 (3x^2 + 2x)dx$

$$= \frac{1}{3}(x^3 + x^2)\Big|_1^4$$
$$= \frac{1}{3}[(4^3 + 4^2) - (1^3 + 1^2)]$$
$$= \frac{1}{3}[(64 + 16) - (1 + 1)]$$
$$= \frac{1}{3}(78) = 26.$$



Mean Value Theorem for Integrals: Let f be continuous on $[a, b]$. Then there exists a point c in (a, b) such that: $\boldsymbol{f(c) = \bar{f} = \frac{1}{b-a}\int_a^b f(t)dt}$.



Proof: Let $F(x) = \int_a^x f(t)dt$.
$F(x)$ is continuous on $[a,b]$ and differentiable on (a,b) and therefore satisfies the Mean Value Theorem.

Thus there exists a c in (a,b) such that

$$F'(c) = \frac{F(b)-F(a)}{b-a} \ .$$

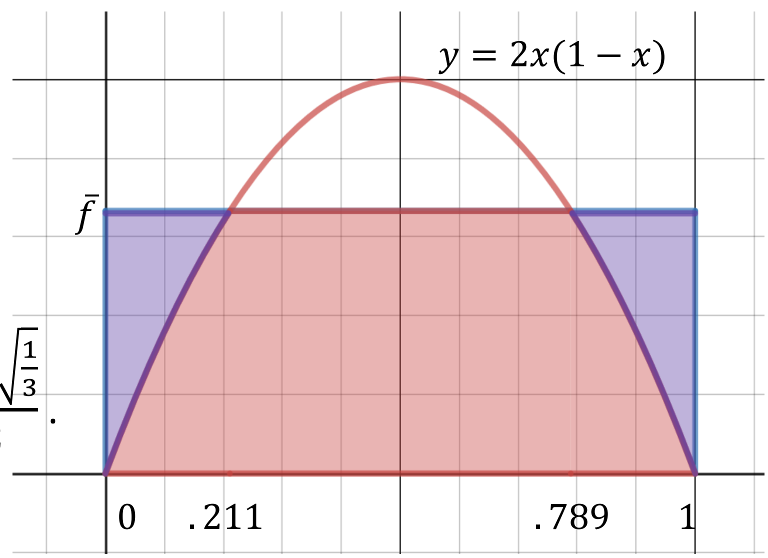
But $F(b) - F(a) = \int_a^b f(t)dt$, and $F'(c) = f(c)$, so

$$f(c) = \frac{1}{b-a}\int_a^b f(t)dt.$$

Ex. Find the point(s) in the interval $(0,1)$ at which $f(x) = 2x(1-x)$ equals its average value on $[0,1]$.

$$\bar{f} = \frac{1}{1-0}\int_0^1 2x(1-x)dx = \left(x^2 - \frac{2x^3}{3}\right)\Big|_0^1 = \frac{1}{3}$$

So we must find a point x in $(0,1)$ such that $f(x) = \frac{1}{3}$.



$$2x(1-x) = \frac{1}{3}$$
$$-2x^2 + 2x = \frac{1}{3}$$
$$2x^2 - 2x + \frac{1}{3} = 0.$$

Using the quadratic formula:

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(2)(\frac{1}{3})}}{2(2)} = \frac{2 \pm \sqrt{\frac{4}{3}}}{4} = \frac{1 \pm \sqrt{\frac{1}{3}}}{2}.$$

So there are 2 points:

$$x = \frac{1-\sqrt{\frac{1}{3}}}{2} \approx 0.211\,, \quad x = \frac{1+\sqrt{\frac{1}{3}}}{2} \approx 0.789.$$

Ex. Find the point(s) in the interval $(0,2)$ at which $f(x) = x^3$ equals its average value on $[0,2]$.

$$\bar{f} = \frac{1}{2-0}\int_0^1 x^3 dx$$
$$= \frac{1}{2}\left(\frac{1}{4}x^4\right) \Big|_0^2$$
$$= \left(\frac{1}{8}x^4\right) \Big|_0^2 = \frac{1}{8}(2^4 - 0^4) = 2$$

So we need to find all points in the interval $(0,2)$ such that:

$$f(x) = x^3 = 2 \quad \Longrightarrow \quad x = \sqrt[3]{2}.$$