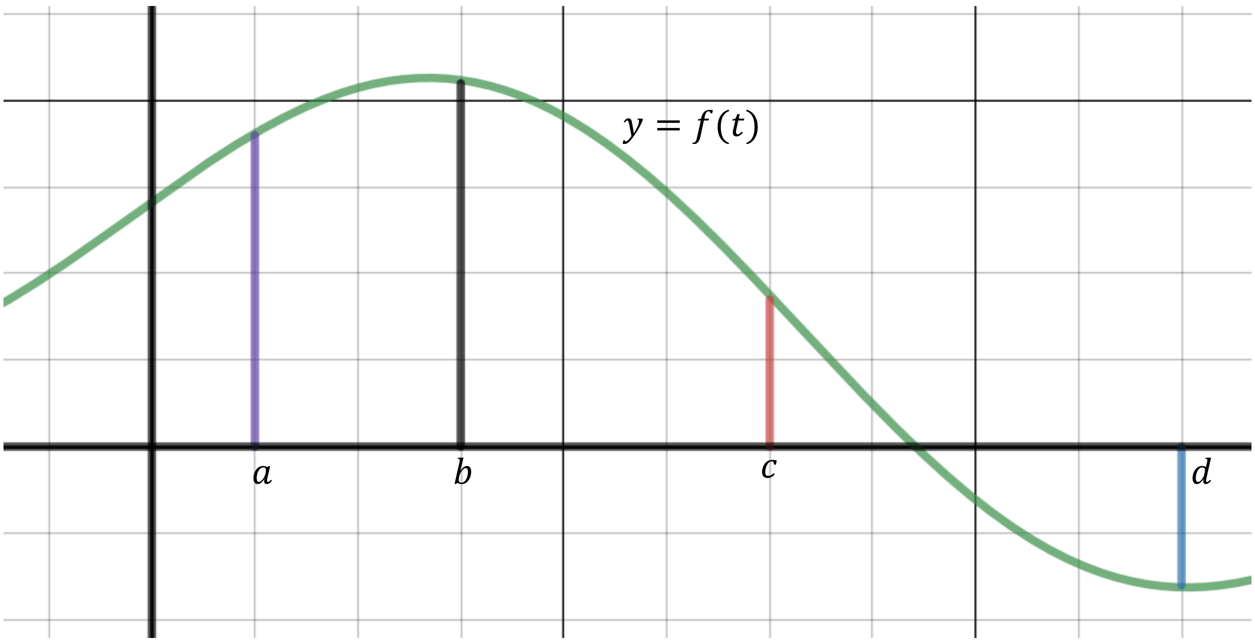
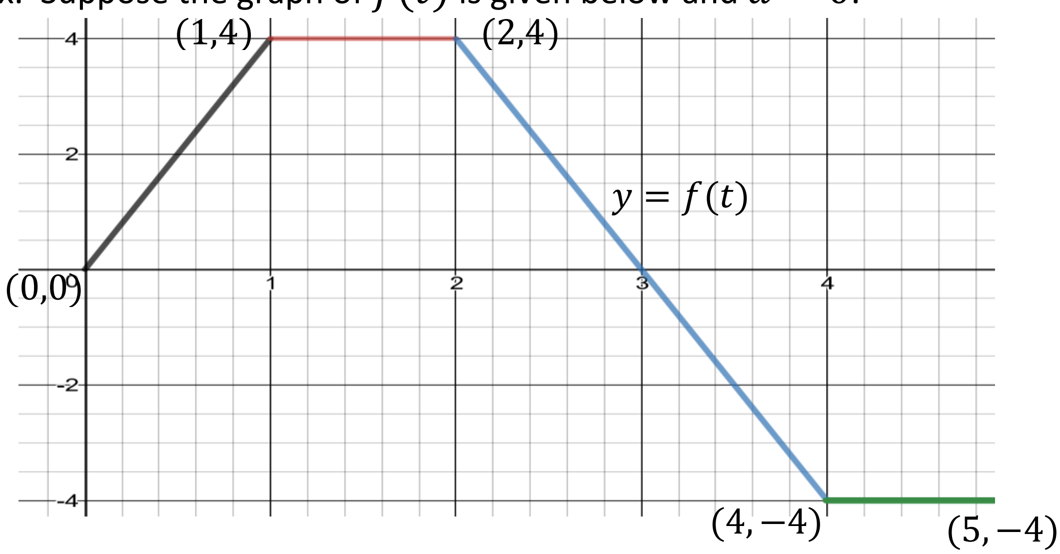
The Fundamental Theorem of Calculus

Let $A(x) = \int_a^x f(t)dt$; $x \geq a$ be the "net area" function for $f(t)$.



$A(b) = \int_a^b f(t)dt$, $\quad A(c) = \int_a^c f(t)dt$, $\quad A(d) = \int_a^d f(t)dt.$

Ex. Suppose the graph of $f(t)$ is given below and $a = 0$.



Find $A(1),\ A(2),\ A(3),\ A(4),\ A(5)$.

$A(1) = \int_0^1 f(t)dt = \frac{1}{2}(1)(4) = 2$

$A(2) = \int_0^2 f(t)dt = 2 + 1(4) = 6$

$A(3) = \int_0^3 f(t)dt = 6 + \frac{1}{2}(1)(4) = 8$

$A(4) = \int_0^4 f(t)dt = 8 - \frac{1}{2}(1)(4) = 6$

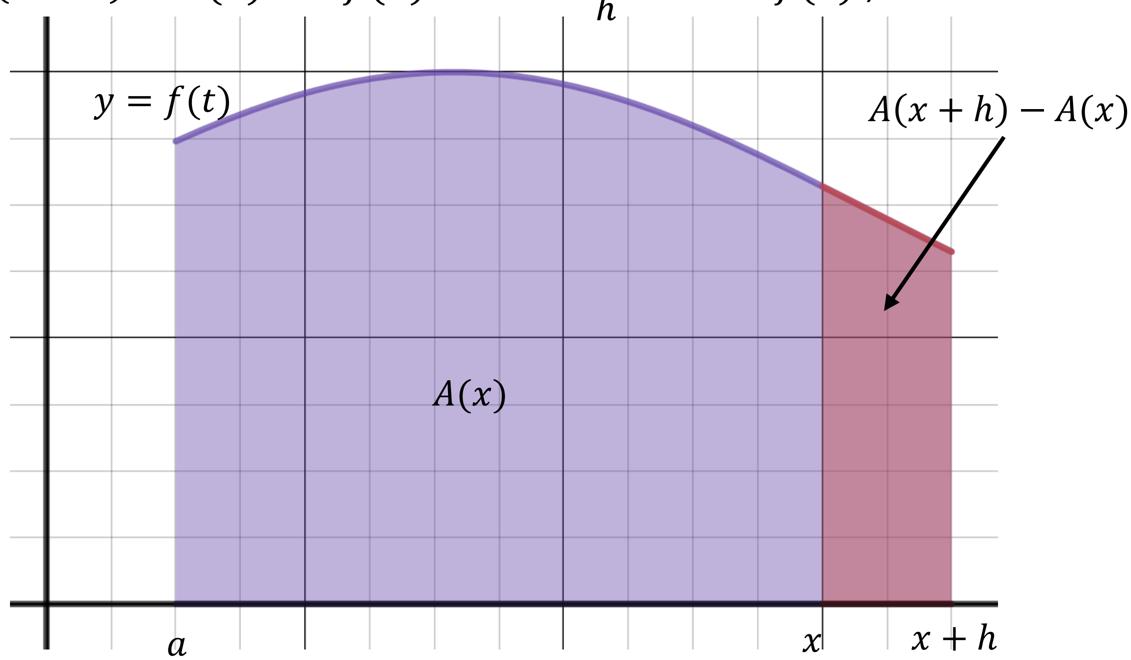
$A(5) = \int_0^5 f(t)dt = 6 - 1(4) = 2.$

Now we want to find $A'(x)$.

$A'(x) = \lim_{h\to 0} \frac{A(x+h)-A(x)}{h}$.

When h is small:

$A(x+h) - A(x) \approx hf(x)$ or $\frac{A(x+h)-A(x)}{h} \approx f(x)$;



so now:

$$\lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \to 0} f(x) = f(x).$$

Thus $A'(x) = \frac{d}{dx} \int_a^x f(t)dt = f(x)$.

Fundamental Theorem of Calculus (part 1)

If f is continuous on $[a, b]$, then the net area function

$$A(x) = \int_a^x f(t)dt \qquad a \leq x \leq b,$$

Is continuous on $[a, b]$ and differentiable on (a, b). The net area function also satisfies

$$A'(x) = \frac{d}{dx} \int_a^x f(t)dt = f(x).$$

Thus the net area function of f is an antiderivative of f on $[a, b]$.

If $F(x)$ is any antiderivative of f, then $F(x) = A(x) + C, \quad a \leq x \leq b.$

Since $A(a) = 0$, we have

$$F(b) - F(a) = (A(b) + C) - (A(a) + C) = A(b) = \int_a^b f(t)dt.$$

Fundamental Theorem of Calculus (part 2)

If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Now we have a much simpler way to evaluate definite integrals (rather than evaluating that complicated limit). We just need to find any antiderivative $F(x)$ of $f(x)$, Evaluate F at b and a and subtract.

Ex. Evaluate

a. $\int_0^3 (x^2 + 2x)dx$

b. $\int_1^2 x\left(\sqrt[3]{x} + \frac{1}{\sqrt{x^5}}\right) dx$

a. To evaluate $\int_0^3 (x^2 + 2x)dx$ we need an antiderivative of $f(x) = x^2 + 2x$. In this case, $F(x) = \frac{1}{3}x^3 + x^2$ works.

$$\int_0^3 (x^2 + 2x)dx = (\tfrac{1}{3}x^3 + x^2)\ |_{x=0}^{x=3}$$

$$= \left(\frac{1}{3}(3)^3 + 3^2\right) - \left(\frac{1}{3}(0)^3 + 0^2\right) = 9 + 9 = 18.$$

b. $\int_1^2 x\left(\sqrt[3]{x} + \frac{1}{\sqrt{x^5}}\right) dx = \int_1^2 x\left(x^{\frac{1}{3}} + x^{-\frac{5}{2}}\right) dx = \int_1^2 (x^{\frac{4}{3}} + x^{-\frac{3}{2}})dx$

To evaluate the last integral we need an antiderivative for $f(x) = x^{\frac{4}{3}} + x^{-\frac{3}{2}}$. In this case , $F(x) = \frac{3}{7}x^{\frac{7}{3}} - 2x^{-\frac{1}{2}}$ works.

$$\int_1^2 (x^{\frac{4}{3}} + x^{-\frac{3}{2}})dx = (\tfrac{3}{7}x^{\frac{7}{3}} - 2x^{-\frac{1}{2}})\ |_{x=1}^{x=2}$$

$$= \left(\frac{3}{7}(2)^{\frac{7}{3}} - 2(2)^{-\frac{1}{2}}\right) - (\tfrac{3}{7}(1)^{\frac{7}{3}} - 2(1)^{-\frac{1}{2}})$$

$$= \frac{3}{7}(2)^{\frac{7}{3}} - \frac{2}{\sqrt{2}} - (\tfrac{3}{7} - 2) = \frac{3}{7}(2)^{\frac{7}{3}} - \sqrt{2} + \frac{11}{7}$$

$$\int_1^2 x\left(\sqrt[3]{x} + \frac{1}{\sqrt{x^5}}\right) dx = \frac{3}{7}(2)^{\frac{7}{3}} - \sqrt{2} + \frac{11}{7}\,.$$

Ex. Evaluate $\int_0^\pi sinxdx$

We need an antiderivative of $f(x) = sinx$. $F(x) = -cosx$ works.

$$\int_0^\pi sinxdx = -\ (cosx)|_0^\pi$$
$$= -[cos\pi - cos0]$$
$$= -[-1 - 1]$$
$$= 2.$$

Ex. Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1+sin^2x}{sin^2x} dx$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1+sin^2x}{sin^2x} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\frac{1}{sin^2x} + \frac{sin^2x}{sin^2x})\, dx\ = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (csc^2x + 1)\, dx$$

So we need to find an antiderivative for $f(x) = csc^2x + 1$.
$F(x) = -cotx + x$ works.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1+sin^2x}{sin^2x} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (csc^2x + 1)\, dx = (-cotx + x|_{\pi/4}^{\pi/2}$$
$$= \left(-\cot\left(\frac{\pi}{2}\right) + \frac{\pi}{2}\right) - (-\cot\left(\frac{\pi}{4}\right) + \frac{\pi}{4})$$
$$= \left(0 + \frac{\pi}{2}\right) - \left(-1 + \frac{\pi}{4}\right) = 1 + \frac{\pi}{4}.$$

The first part of the Fundamental theorem of Calculus allows us to take derivatives of "net area" functions. That is, functions where the unknown is one (or both) endpoints of a definite integral.

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Ex. Find the derivatives of the following functions:

a. $g(x) = \int_{-2}^{x} \sqrt{1+t^4}dt$

b. $h(x) = \int_5^x sin^2\left(\frac{\pi t^2}{2}\right) dt$

c. $g(x) = \int_x^{-2} \sqrt{1+t^4}dt$

d. $h(x) = \int_0^{x^3} (sect)dt$

e. $g(x) = \int_{sinx}^{\pi} \sqrt[4]{1+t^2}dt$

a. $g'(x) = \frac{d}{dx}\int_{-2}^{x} \sqrt{1+t^4}dt = \sqrt{1+x^4};$ since $f(t) = \sqrt{1+t^4}$

b. $h'(x) = \frac{d}{dx}\int_5^x sin^2\left(\frac{\pi t^2}{2}\right) dt = sin^2\left(\frac{\pi x^2}{2}\right);$ $f(t) = sin^2\left(\frac{\pi t^2}{2}\right)$

c. $g(x) = \int_x^{-2} \sqrt{1+t^4}dt = -\int_{-2}^{x} \sqrt{1+t^4}dt;$

So $g'(x) = -\sqrt{1+x^4}.$

d. $h(x) = \int_0^{x^3} (sect)dt;$ Let $u = x^3$, so $y = h(u) = \int_0^u (sect)dt$

By the chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (secu)(3x^2)$

$= (\sec(x^3))(3x^2);$ so

$h'(x) = (\sec(x^3))(3x^2).$

e. $g(x) = \int_{sinx}^{\pi} \sqrt[4]{1 + t^2 dt} = -\int_{\pi}^{sinx} \sqrt[4]{1 + t^2} dt;$

Let $u = sinx;$

$y = g(u) = -\int_{\pi}^{u} \sqrt[4]{1 + t^2} dt$

By the chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\left(\sqrt[4]{1 + u^2}\right)(cosx)$

$= -\left(\sqrt[4]{1 + sin^2 x}\right)(cosx)$;

So $g'(x) = -\left(\sqrt[4]{1 + sin^2 x}\right)(cosx)$.