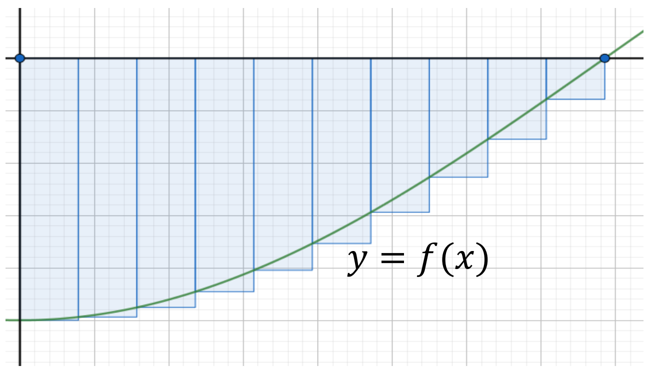
Definite Integrals

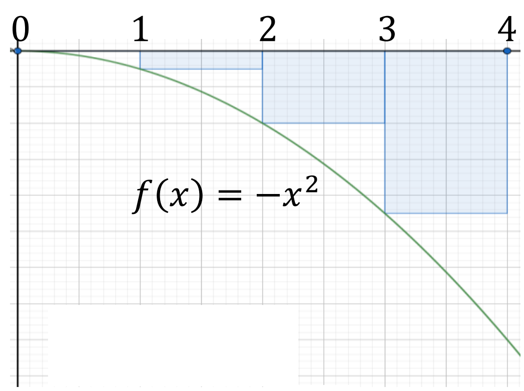
So far we have only considered Riemann sums for functions $f(x)$ over intervals where $f(x) \geq 0$. What happens if $f(x) \leq 0$ on an interval or if $f(x) \geq 0$ on part of the interval and $f(x) \leq 0$ on the rest of the interval?

If $f(x) \leq 0$ on an interval $[a, b]$ then the Riemann sums will approximate minus the area trapped between the graph and the x-axis.



Ex. Calculate the left Riemann sum for $f(x) = -x^2$ on $[0,4]$ using 4 subdivisions.

$$n = 4, \quad \Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$$



Left Riemann sum=

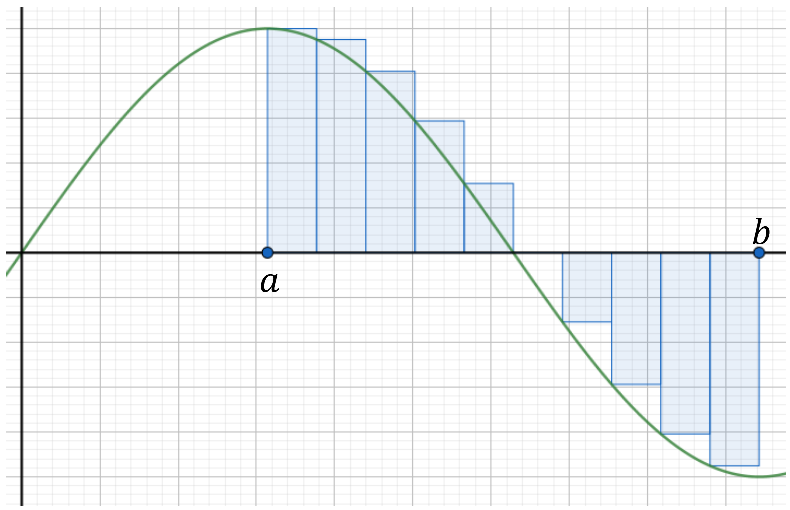
$$= (f(0))(1) + (f(1))(1) + (f(2))(1) + (f(3))(1)$$

$$= -0^2(1) - (1^2)(1) - (2^2)(1) - (3^2)(1)$$

$$= 0 - 1 - 4 - 9 = -14.$$

This is an approximation of minus the area between the graph of $f(x)$ and the x axis. The more subdivisions we take the better the approximation will be. In the limit, we will get minus the area between the graph of $f(x)$ and the x axis.

If $f(x)$ is sometimes positive and sometimes negative on an interval $[a, b]$ the Riemann sums will approximate the net area between the graph of $f(x)$ and the x axis, counting the area with a positive sign when $f(x) \geq 0$ and with a negative sign when $f(x) \leq 0$.



Net Area= $\lim_{n\to\infty} \sum_{k=1}^{n} \mathrm{f}(x_k{}^*)\Delta x$.

Def. A **general partition** of $[a, b]$ consists of n subintervals

$$[x_{0,}x_1], [x_{1,}x_2], [x_{2,}x_3], \ldots, [x_{n-1,}x_n]$$

where $x_0 = a$ and $x_n = b$. The length of the kth subinterval is $\Delta x_k = x_k - x_{k-1}$, for $k = 1,2, \ldots, n$.

Def. a **General Riemann Sum** for f on $[a, b]$ is the sum

$$\sum_{k=1}^{n} \mathrm{f}(x_k{}^*)\Delta x_k = \mathrm{f}(x_1{}^*)\Delta x_1 + \mathrm{f}(x_2{}^*)\Delta x_2 + \cdots + \mathrm{f}(x_n{}^*)\Delta x_n$$

where $x_k{}^*$ is any point in $[x_{k-1,}x_k]$.

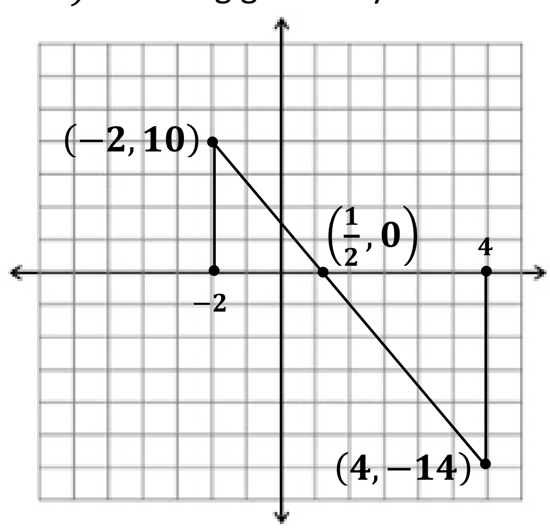
Def. A function f on $[a,b]$ is **integrable** on $[a,b]$ if $\lim_{max\Delta x_k \to 0} \sum_{k=1}^{n} f({x_k}^*)\Delta x_k$ exists and is unique over all partitions of $[a,b]$ and all choices of ${x_k}^*$ on a partition. This limit is the **definite integral of f from a to b**, written

$$\int_a^b f(x)dx = \lim_{max\Delta x_k \to 0} \sum_{k=1}^{n} f({x_k}^*)\Delta x_k.$$

Please remember the difference between an indefinite integral and a definite integral. The indefinite integral of f, $\int f(x)dx$, is a family of functions (the antiderivatives of $f(x)$). The definite integral of f from a to b, $\int_a^b f(x)dx$, is a real number (the net area of a region between $f(x)$ and the x axis).

Because $\int_a^b f(x)dx =$Net area under the graph of $f(x)$, we can calculate some definite integrals by just using geometry.

Ex. Evaluate $\int_{-2}^{4}(2-4x)\,dx$ using geometry.

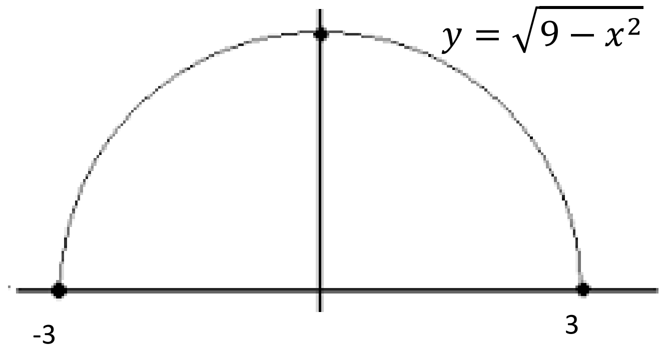


$\int_{-2}^{4}(2-4x)\,dx =$ (area of upper triangle)-(area of lower triangle)

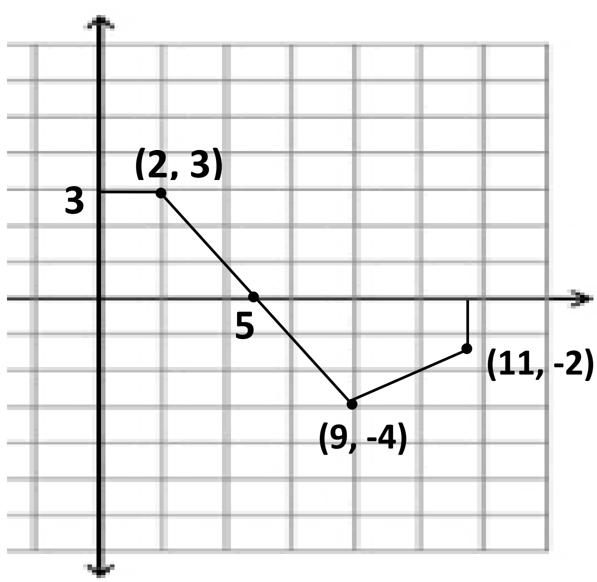
$$= \left(\frac{1}{2}\right)\left(\frac{5}{2}\right)(10) - \left(\frac{1}{2}\right)\left(\frac{7}{2}\right)(14) = \frac{25}{2} - \frac{49}{2} = -12.$$

Ex. Evaluate $\int_{-3}^{3} \sqrt{9-x^2}\,dx$ using geometry.

$\int_{-3}^{3} \sqrt{9-x^2}\,dx =$ area of semi-circle or radius $3 = \frac{1}{2}\pi(3)^2 = \frac{9\pi}{2}$.



Ex. Using geometry evaluate $\int_{0}^{11} f(x)dx$ given the graph of $f(x)$ below.

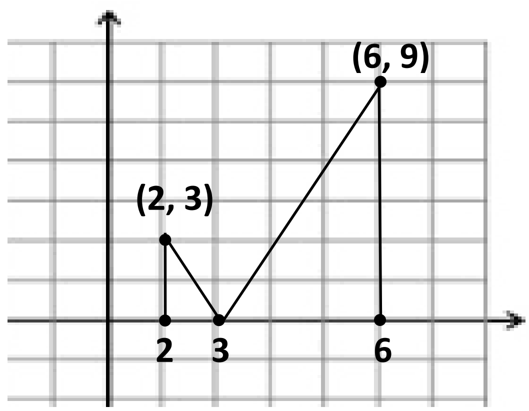


$$\int_0^{11} f(x)dx = \int_0^2 f(x)dx + \int_2^5 f(x)dx + \int_5^9 f(x)dx + \int_9^{11} f(x)dx$$

$$= 2(3) + \frac{1}{2}(3)(3) - \frac{1}{2}(4)(4) - [2(2) + \frac{1}{2}(2)(2)]$$

$$= 6 + \frac{9}{2} - 8 - 6 = -\frac{7}{2}.$$

Ex. Using geometry evaluate $\int_2^6 |3x-9|dx$.



$$\int_2^6 |3x-9|dx = \frac{1}{2}(1)(3) + \frac{1}{2}(3)(9)$$

$$= \frac{3}{2} + \frac{27}{2} = 15.$$

Theorem: If f is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuities, then f is integrable on $[a, b]$

(i.e., $\lim\limits_{max\Delta x_k \to 0} \sum_{k=1}^{n} \mathrm{f}(x_k{}^*)\Delta x_k$ uniquely exists.)

Properties of Definite Integrals: (Assume f is integrable on $[a, b]$)

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
2. $\int_a^a f(x)dx = 0$
3. $\int_a^b \big(f(x) \pm g(x)\big)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
4. $\int_a^b cf(x)dx = c\int_a^b f(x)dx;$ where c is a constant
5. $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$
6. $|f|$ is integrable on $[a, b]$, and

   $\int_a^b |f(x)|dx =$ sum of areas of regions bounded by graph of f and x axis on $[a, b]$.

Ex . Suppose $\int_2^8 f(x)dx = 12$ and $\int_6^8 f(x)dx = 2$, find

1. $\int_2^6 f(x)dx$
2. $\int_8^6 f(x)dx$
3. $\int_6^8 3f(x)dx$
4. $\int_8^2 -2f(x)dx$.

1. $\int_2^6 f(x)dx + \int_6^8 f(x)dx = \int_2^8 f(x)dx$

$$\int_2^6 f(x)dx + 2 = 12$$

$$\int_2^6 f(x)dx = 10.$$

2. $\int_8^6 f(x)dx = -\int_6^8 f(x)dx$

$$\int_8^6 f(x)dx = -2.$$

3. $\int_6^8 3f(x)dx = 3\int_6^8 f(x)dx = 3(2) = 6.$

4. $\int_8^2 -2f(x)dx = -2\int_8^2 f(x)dx = 2\int_2^8 f(x)dx = 2(12) = 24.$

Ex. Suppose $\int_{-3}^{6} f(x)dx = 20$ and $\int_{3}^{-3} f(x)dx = 12$, find

a. $\int_{3}^{6} f(x)dx$

b. $\int_{6}^{-3} 3f(x)dx$.

a. $\int_{-3}^{3} f(x)dx + \int_{3}^{6} f(x)dx = \int_{-3}^{6} f(x)dx$

$\int_{3}^{-3} f(x)dx = 12 \implies \int_{-3}^{3} f(x)dx = -12$

$-12 + \int_{3}^{6} f(x)dx = 20$

$\int_{3}^{6} f(x)dx = 32$

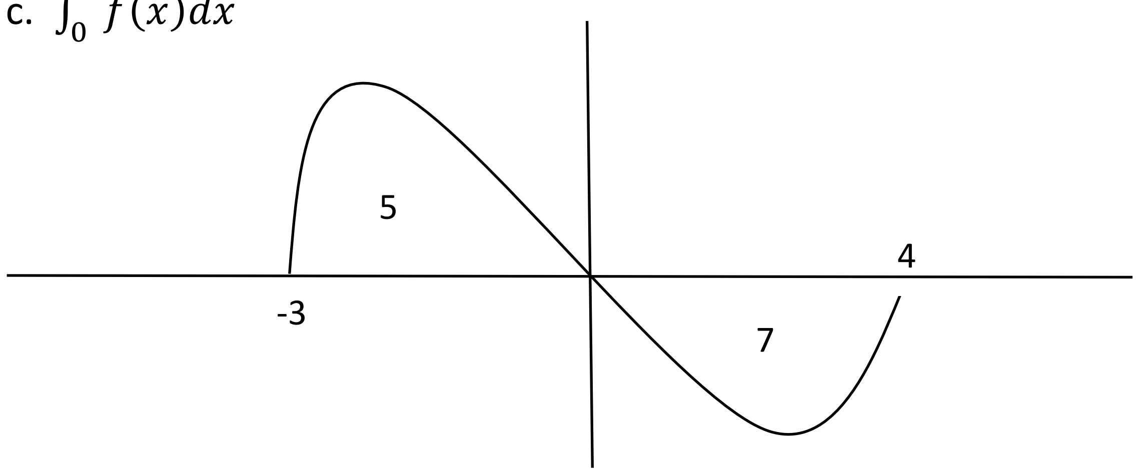
b. $\int_{6}^{-3} 3f(x)dx = 3\int_{6}^{-3} f(x)dx = -3\int_{-3}^{6} f(x)dx = -3(20) = -60$

Ex. A graph of $f(x)$ is given below with the areas of each region. Find

a. $\int_{-3}^{4} f(x)dx$

b. $\int_{-3}^{4} |f(x)|dx$

c. $\int_{0}^{4} f(x)dx$



a. $\int_{-3}^{4} f(x)dx = 5 - 7 = -2$

b. $\int_{-3}^{4} |f(x)|dx = 5 + 7 = 12$

c. $\int_{0}^{4} f(x)dx = -7.$

Using the Limit definition of the definite integral:

$$\int_a^b f(x)dx = \lim_{max\Delta x_k \to 0} \sum_{k=1}^{n} \mathrm{f}(x_k{}^*)\Delta x_k \ .$$

Ex. Identify which definite integral is represented by the following limits

a. $\lim_{max\Delta x_k \to 0} \sum_{k=1}^{n} (4(x_k{}^*)^5 - 3(x_k{}^*)^3 + 2)\Delta x_k$; on $[-1,5]$.

b. $\lim_{max\Delta x_k \to 0} \sum_{k=1}^{n} (2x_k{}^* \sin(x_k{}^*) - x_k{}^*)\Delta x_k$; on $[0, \frac{\pi}{2}]$.

a. $\int_{-1}^{5} (4x^5 - 3x^3 + 2)dx$

b. $\int_0^{\frac{\pi}{2}} (2xsinx - x)dx.$

Ex. Calculate $\int_0^3 (x^2 + 2x)dx$ using the limit definition of the definite integral. Use right endpoints and subintervals of equal length.

Since all of the subintervals are the same length, all of the Δx_k are the same size (Δx). In fact, $\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$.

${x_k}^*$ is the right endpoint of the kth interval so:

$${x_k}^* = a + k\Delta x = 0 + k\left(\frac{3}{n}\right) = \frac{3k}{n}; \quad f(x) = x^2 + 2x.$$

$$\int_a^b f(x)dx = \lim_{max\Delta x_k \to 0} \sum_{k=1}^n \mathrm{f}({x_k}^*)\Delta x_k$$

$$= \lim_{n\to\infty} \sum_{k=1}^n \mathrm{f}(a + k\Delta x)\Delta x = \lim_{n\to\infty} \sum_{k=1}^n \mathrm{f}\left(\frac{3k}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^n \left[\left(\frac{3k}{n}\right)^2 + 2\left(\frac{3k}{n}\right)\right]\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n}\sum_{k=1}^n \left[\frac{9k^2}{n^2} + \frac{6k}{n}\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left(\frac{9}{n^2}\sum_{k=1}^n k^2 + \frac{6}{n}\sum_{k=1}^n k\right)$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[\frac{9}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{6}{n}\left(\frac{n(n+1)}{2}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{18}{n^2}\left(\frac{n(n+1)}{2}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{27}{n^3}\left(\frac{2n^3+3n^2+n}{6}\right) + \frac{18}{n^2}\left(\frac{n^2+n}{2}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{27}{6}\left(\frac{2n^3+3n^2+n}{n^3}\right) + \frac{18}{2}\left(\frac{n^2+n}{n^2}\right)\right]$$

$$= \frac{27}{6}(2) + \frac{18}{2}(1) = 9 + 9 = 18.$$