

Antiderivatives

Def. A function F is an **antiderivative** of f on an interval I provided

$$F'(x) = f(x), \text{ for all } x \text{ in } I.$$

Antiderivatives will become very important when we talk about integration.

If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, i.e. $F'(x) = f(x)$ and $G'(x) = f(x)$, Then $G'(x) = F'(x)$. As we saw from the Mean Value Theorem, this means that $G(x) = F(x) + C$, where C is a constant.

So an antiderivative of a function is not unique, but any antiderivative of a function f differs from any other antiderivative of f by a constant.

So far we have developed a number of formulas for derivatives of functions. To find an antiderivative of a function $f(x)$, we need to go “backwards”. That is we need to find a function $F(x)$ such that $F'(x) = f(x)$.

Ex. Find all antiderivatives of $f(x) = 1$.

This one isn't so bad. We have to find a function $F(x)$ such that $F'(x) = 1$. If we take $F(x) = x$, that will work. So $F(x) = x$ is an antiderivative of $f(x) = 1$. If we want all antiderivatives of $f(x) = 1$, we take $F(x) = x + C$.

Ex. Find all antiderivatives of $f(x) = x$.

So we need a function $F(x)$ such that $F'(x) = x$. If we take $F(x) = \frac{1}{2}x^2$ we will have an antiderivative of $f(x) = x$. To get all antiderivatives we take $F(x) = \frac{1}{2}x^2 + C$.

Ex. Find all antiderivatives of $h(x) = x^n$, where n is a positive integer.

We know when we take a derivative of x^n , n a positive integer, we get

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

So if we want to go “backwards” let’s try taking as an antiderivative of x^n ,

$$H(x) = \frac{1}{n+1}x^{n+1}.$$

Notice that if n is a positive integer (or even a positive rational number) that

$$H'(x) = x^n.$$

So all of the antiderivatives of $h(x) = x^n$, n a positive integer, are given by

$$H(x) = \frac{1}{n+1}x^{n+1} + C.$$

In particular, when $n = 3$, all of the antiderivatives of $g(x) = x^3$ are given by

$$G(x) = \frac{1}{4}x^4 + C.$$

When $n = 2$, all of the antiderivatives of $f(x) = x^2$ are given by

$$F(x) = \frac{1}{3}x^3 + C.$$

Notice that if $F(x)$ is an antiderivative of $f(x)$, then $kF(x)$ is an antiderivative of $kf(x)$, where k is a constant.

Ex. Find all of the antiderivatives of $f(x) = -9x^5$ and $g(x) = \frac{x^8}{3}$.

We know that an antiderivative of x^5 is given by $\frac{1}{6}x^6$ so an antiderivative of $-9x^5$ is given by $-9\left(\frac{1}{6}x^6\right) = -\frac{3}{2}x^6$. Thus all of the antiderivatives of $f(x) = -9x^5$ are given by $F(x) = -\frac{3}{2}x^6 + C$.

We know that an antiderivative of x^8 is given by $\frac{1}{9}x^9$ so an antiderivative of $\frac{x^8}{3} = \frac{1}{3}x^8$ is given by $\frac{1}{3}\left(\frac{1}{9}x^9\right) = \frac{1}{27}x^9$. Thus all of the antiderivatives of $g(x) = \frac{x^8}{3}$ are given by $G(x) = \frac{1}{27}x^9 + C$.

The symbol we use for the antiderivatives of a function $f(x)$ is what's called the **indefinite integral of $f(x)$** , written $\int f(x)dx$.

So we would say $\int -9x^5 dx = -\frac{3}{2}x^6 + C$

Power Rule for Indefinite Integrals

$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$ where $n \neq -1$ is a real number and C is any constant.

This follows from the fact that
$$\frac{d}{dx} \left(\frac{1}{n+1} x^{n+1} + C \right) = \frac{d}{dx} \left(\frac{1}{n+1} x^{n+1} \right) + \frac{d}{dx} (C)$$

$$= x^n.$$

Note: I'm "cheating" a bit here since we only know the derivative power rule $\frac{d}{dx} (x^n) = nx^{n-1}$ when n is a rational number. However, this rule as well as the Power Rule for indefinite integrals is true for n a real number.

Just as in the case with taking derivatives, the indefinite integral of the sum or difference of 2 functions is the sum or difference of the indefinite integrals. The indefinite integral of a constant times a function is that constant times the indefinite integral of the function.

Constant Multiple and Sum/Difference Rules

$$\int cf(x)dx = c \int f(x)dx$$

$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx.$$

Ex. Find the following indefinite integrals (Try to first write the function you are integrating, called the integrand, as a sum of terms that are just constants times powers of the variable)

a. $\int (2x^3 - 4 + \frac{6}{\sqrt{x}}) dx$

b. $\int \left(\frac{2x^{11} - 3x^{-5}}{x^5} \right) dx$

c. $\int (y^2 + 3)(y^3 - 1) dy$

$$\begin{aligned} \text{a. } \int (2x^3 - 4 + \frac{6}{\sqrt{x}}) dx &= \int (2x^3 - 4 + 6x^{-\frac{1}{2}}) dx \\ &= 2 \int x^3 dx - 4 \int 1 dx + 6 \int x^{-\frac{1}{2}} dx \\ &= 2 \left(\frac{1}{4} x^4 \right) - 4x + 6 \left(\frac{1}{\frac{1}{2}} x^{\frac{1}{2}} \right) + C \\ &= \frac{1}{2} x^4 - 4x + 6 \left(2x^{\frac{1}{2}} \right) + C \\ &= \frac{1}{2} x^4 - 4x + 12 \left(x^{\frac{1}{2}} \right) + C \end{aligned}$$

$$\begin{aligned} \text{b. } \int \left(\frac{2x^{11} - 3x^{-5}}{x^5} \right) dx &= \int \left(2 \frac{x^{11}}{x^5} - 3 \frac{x^{-5}}{x^5} \right) dx \\ &= \int (2x^6 - 3x^{-10}) dx \\ &= 2 \int x^6 dx - 3 \int x^{-10} dx \\ &= 2 \left(\frac{1}{7} x^7 \right) - 3 \left(\frac{1}{-9} \right) x^{-9} + C \\ &= \frac{2}{7} x^7 + \frac{1}{3} x^{-9} + C \end{aligned}$$

$$\begin{aligned} \text{c. } \int (y^2 + 3)(y^3 - 1) dy &= \int (y^5 - y^2 + 3y^3 - 3) dy \\ &= \int y^5 dy - \int y^2 dy + 3 \int y^3 dy - 3 \int 1 dy \\ &= \frac{1}{6} y^6 - \frac{1}{3} y^3 + 3 \left(\frac{1}{4} y^4 \right) - 3y + C \\ &= \frac{1}{6} y^6 + \frac{3}{4} y^4 - \frac{1}{3} y^3 - 3y + C. \end{aligned}$$

Indefinite Integrals of Trig Functions:

A Table of derivatives can help us develop a table of antiderivatives

1. $\frac{d}{dx}(\sin ax) = a \cos ax$	$\int \cos ax \, dx = \frac{1}{a} \sin(ax) + C$
2. $\frac{d}{dx}(\cos ax) = -a \sin ax$	$\int \sin ax \, dx = -\frac{1}{a} \cos(ax) + C$
3. $\frac{d}{dx}(\tan ax) = a \sec^2 ax$	$\int \sec^2 ax \, dx = \frac{1}{a} \tan(ax) + C$
4. $\frac{d}{dx}(\cot ax) = -a \csc^2 ax$	$\int \csc^2 ax \, dx = -\frac{1}{a} \cot(ax) + C$
5. $\frac{d}{dx}(\sec ax) = a(\sec ax)(\tan ax)$	$\int (\sec ax)(\tan ax) \, dx = \frac{1}{a} \sec(ax) + C$
6. $\frac{d}{dx}(\csc ax) = -a(\csc ax)(\cot ax)$	$\int (\csc ax)(\cot ax) \, dx = -\frac{1}{a} \csc(ax) + C$

Ex. Evaluate the following indefinite integrals

- $\int \cos(6x) \, dx$
- $\int \csc^2(3x) \, dx$
- $\int (2 \sin(4x) + 6 \sec^2(\frac{x}{3})) \, dx$

a. $\int \cos(6x) \, dx = \frac{1}{6} \sin(6x) + C$

b. $\int \csc^2(3x) \, dx = -\frac{1}{3} \cot(3x) + C$

c. $\int \left(2 \sin(4x) + 6 \sec^2 \left(\frac{x}{3} \right) \right) \, dx$

$$= 2 \int \sin(4x) \, dx + 6 \int \sec^2 \left(\frac{x}{3} \right) \, dx$$

$$= 2 \left(-\frac{1}{4} \cos(4x) \right) + 6 \left(3 \tan \left(\frac{x}{3} \right) \right) + C$$

$$= -\frac{1}{2} \cos(4x) + 18 \tan \left(\frac{x}{3} \right) + C.$$

Ex. Evaluate $\int [3 \sec(x) \tan(x) - 2 \csc(5x) \cot(5x)] dx$

$$\begin{aligned} & \int [3 \sec(x) \tan(x) - 2 \csc(5x) \cot(5x)] dx \\ &= 3 \int \sec(x) \tan(x) dx - 2 \int \csc(5x) \cot(5x) dx \\ &= 3 \sec(x) + 2 \left(\frac{1}{5}\right) \csc(5x) + C \\ &= 3 \sec(x) + \frac{2}{5} \csc(5x) + C. \end{aligned}$$

Differential Equations

A **Differential Equation** is an equation that involves the derivative (or derivatives) of a function as well as possibly the function itself. A solution to a differential equation is a function that satisfies the equation. We have already solved some elementary differential equations when we found antiderivatives.

Ex. Solve the following differential equation

$$f'(x) = 3x^2 + 7.$$

So in this case we are looking for all the antiderivatives of $3x^2 + 7$. In other words we want $\int (3x^2 + 7) dx = x^3 + 7x + C$.

So any function of the form $f(x) = x^3 + 7x + C$ is a solution to this differential equation.

In many cases of differential equations, they are accompanied by an **Initial Condition**. An initial condition is a condition on the unknown function f , that will allow us to calculate what the constant term C is.

Ex. Solve the following differential equation

$$f'(x) = 3x^2 + 7$$

with the initial condition that $f(1) = 4$.

In the previous example we found the **general solution** to this differential equation: $f(x) = x^3 + 7x + C$.

Now we know that $f(1) = 4$, so we can't use just any value of the constant C .

$$4 = f(1) = 1^3 + 7(1) + C = 8 + C \quad \Rightarrow \quad C = -4.$$

So $C = -4$ and the solution to this **Initial Value Problem** (i.e. a differential equation together with an initial condition(s)) is

$$f(x) = x^3 + 7x - 4.$$

Ex. Solve the differential equation: $f'(x) = 9\sin(3x) + 8x^3$ with $f(0) = 5$.

$$\begin{aligned} f(x) &= \int (9\sin(3x) + 8x^3) dx \\ &= 9 \int \sin(3x) dx + 8 \int x^3 dx \\ &= 9 \left(-\frac{1}{3} \cos(3x) \right) + 8 \left(\frac{1}{4} x^4 \right) + C = -3 \cos(3x) + 2x^4 + C. \end{aligned}$$

$$f(0) = 5; \quad \text{so}$$

$$5 = f(0) = -3 \cos(0) + 2(0)^4 + C$$

$$5 = -3 + C \quad \Rightarrow \quad C = 8.$$

$$f(x) = -3 \cos(3x) + 2x^4 + 8.$$

Linear Motion

Initial Value Problems for Velocity and Position.

Suppose an object moves in a line with velocity $v(t)$, $t \geq 0$. Then its position can be found by solving the initial value problem

$$s'(t) = v(t), \quad s(0) = s_0, \quad \text{where } s_0 \text{ is known.}$$

If the acceleration of the object $a(t)$ is given, then the velocity can be found by solving the initial value problem

$$v'(t) = a(t), \quad v(0) = v_0, \quad \text{where } v_0 \text{ is known.}$$

Ex. Runner A begins at point $s(0) = 0ft$ with a velocity of $v(t) = (2t + 1)ft/sec$. Runner B begins with a head start at the point $S(0) = 12ft$ with a velocity of $V(t) = 5ft/sec$. Find the positions of the runners for $t \geq 0$ and determine when runner A overtakes runner B.

For runner A: $s'(t) = v(t) = 2t + 1$, $s(0) = 0$.

$$\text{So } s(t) = \int (2t + 1)dt = t^2 + t + C$$

$$0 = s(0) = 0^2 + 0 + C, \quad \text{thus } C = 0.$$

$$\text{So } s(t) = t^2 + t.$$

For runner B: $S'(t) = V(t) = 5$, $S(0) = 12$

$$S(t) = \int 5dt = 5t + C$$

$$12 = S(0) = 5(0) + C, \quad \text{thus } C = 12$$

$$\text{So } S(t) = 5t + 12.$$

To find when Runner A overtakes Runner B solve $t^2 + t = 5t + 12$.

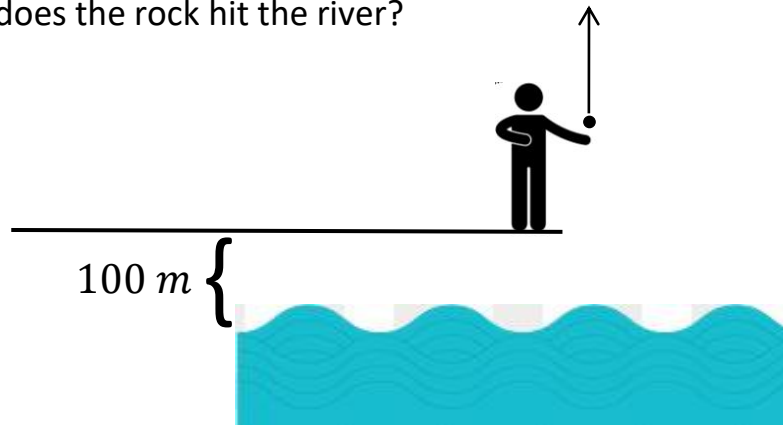
$$t^2 - 4t - 12 = 0$$

$$(t - 6)(t + 2) = 0$$

$$t = 6, -2 \text{ since } t \geq 0, t = 6 \text{ sec.}$$

Ex. Neglecting air resistance the Earth's gravitational pull accelerates objects down toward the Earth at a constant acceleration of $a(t) = -9.8 \text{ m/s}^2$. A rock is thrown vertically upward at $t = 0$ with a velocity of 30 m/s from the edge of a cliff that is 100 m above a river.

- Find the velocity $v(t)$, $t \geq 0$.
- Find the position $s(t)$, $t \geq 0$.
- Find the maximum height of the rock.
- At what speed does the rock hit the river?



$$\text{a. } v'(t) = a(t) = -9.8, \quad v(0) = 30$$

$$v(t) = \int -9.8 dt = -9.8t + C$$

$$30 = v(0) = -9.8(0) + C, \quad C = 30.$$

$$v(t) = -9.8t + 30.$$

b. $s'(t) = v(t) = -9.8t + 30$, $s(0) = 100$

$$s(t) = \int(-9.8t + 30)dt = -4.9t^2 + 30t + C$$

$$100 = s(0) = -4.9(0^2) + 30(0) + C, \quad C = 100.$$

$$s(t) = -4.9t^2 + 30t + 100.$$

c. Maximum height occurs when the velocity is 0.

$$0 = -9.8t + 30 \quad \text{or} \quad t = \frac{30}{9.8} \approx 3.06s.$$

$$s(3.06) = -4.9(3.06^2) + 30(3.06) + 100 \approx 145.92m.$$

d. We have to solve for the time when the rock hits the water.

$$-4.9t^2 + 30t + 100 = 0 \quad \text{(Use the quadratic formula)}$$

$$t = \frac{-30 \pm \sqrt{30^2 - 4(-4.9)(100)}}{2(-4.9)} \approx 8.52s \quad \text{(the other solution is negative)}$$

$$v(8.52) = -9.8(8.52) + 30 \approx -53.50m/s \quad \text{(- means downward)}$$

$$\text{Speed} = |\text{velocity}| = 53.50m/s.$$