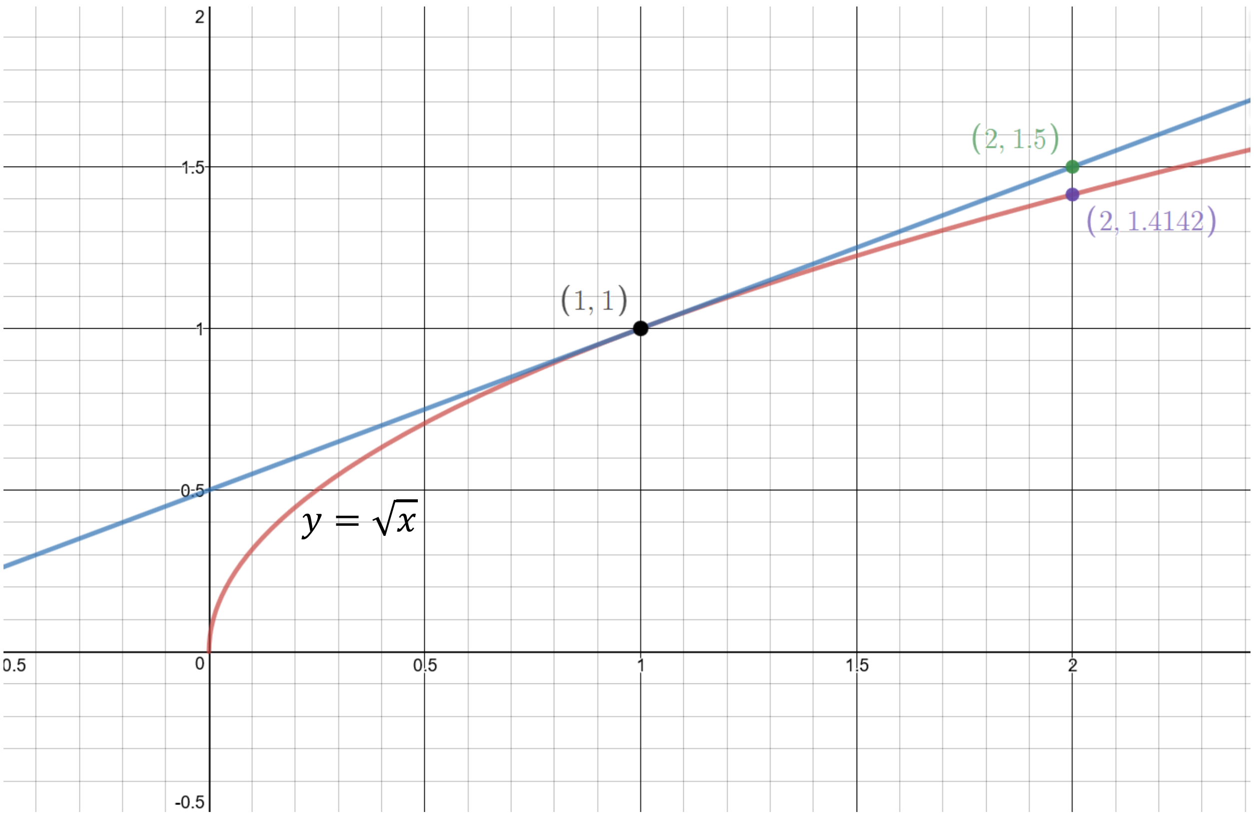
Linear Approximations and Differentials

Suppose we want to approximate $\sqrt{2}$. Since we know the value of $\sqrt{1}$ we can use the tangent line to $y = \sqrt{x}$ at the point $(1,1)$ to approximate $\sqrt{2}$.



To do this we need to find the equation of the tangent line at $(1,1)$ and then find the y value along the tangent line when $x = 2$.

$$f(x) = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Slope of tangent line at $x = 1$ is

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

Equation of tangent line at $x = 1$:

$$y - 1 = \frac{1}{2}(x - 1) \quad \text{or} \quad y = \frac{1}{2}(x - 1) + 1$$

$L(x) = \frac{1}{2}(x - 1) + 1$ is the **linear approximation of** $f(x) = x^{\frac{1}{2}}$ at $x = 1$.

So we can approximate $\sqrt{2}$ by:

$$\sqrt{2} \approx L(2) = \frac{1}{2}(2 - 1) + 1 = \frac{1}{2}(1) + 1 = 1.5.$$

We can define the **error** in the linear approximation by:

actual value=approximate value +**error**, or equivalently:
error=actual value-approximate value.

The **absolute error** is then defined as:

Absolute error=|error|=|actual value-approximate value|.

The **absolute error** in our example is $|f(2) - L(2)| = \left|\sqrt{2} - 1.5\right|$

$$\approx |1.4142 - 1.5| = 0.0858.$$

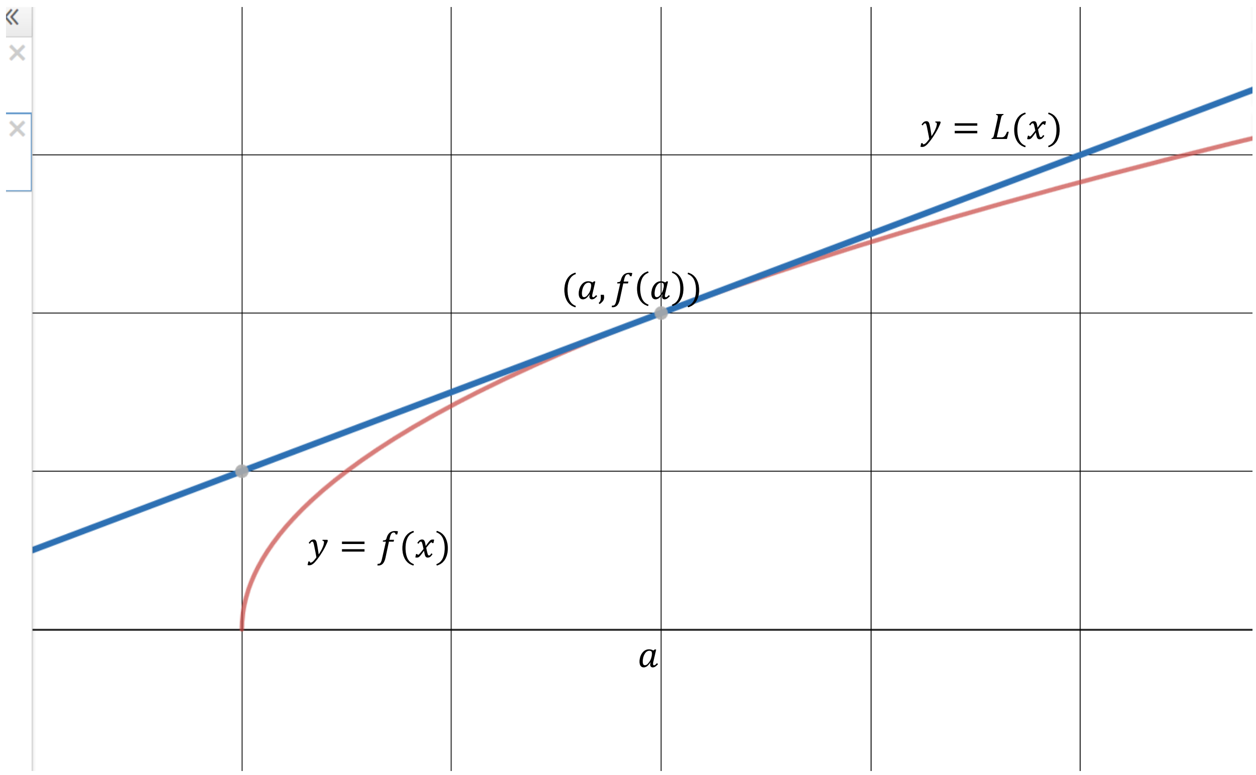
The **percentage error** is $\frac{error}{actual\ value} = \frac{-0.0858}{1.4142} \approx -0.0607 = -0.6.07\%.$

If we wanted to approximate $\sqrt{0.5}$, then put $x = 0.5$ into $L(x)$

$$\sqrt{0.5} \approx L(0.5) = \frac{1}{2}(0.5 - 1) + 1 = \frac{1}{2}(-0.5) + 1 = 0.75.$$

Def. Suppose f is differentiable on an interval I containing the point a. The linear approximation to f at a is the linear function

$$\boldsymbol{L(x) = f(a) + f'(a)(x - a)} \quad \text{for all } x \text{ in } I.$$



$$m = f'(a)$$

Equation of tangent line at $(a, f(a))$:

$$y - f(a) = f'(a)(x - a)$$

$$y = f(a) + f'(a)(x - a).$$

$y = L(x)$ is just the equation of the tangent line to $f(x)$ at $x = a$.

Ex. Find $L(x)$ for the function $y = \sqrt[3]{x}$ at the point $a = 8$. Then use $L(x)$ to approximate the values of $\sqrt[3]{8.5}$ and $\sqrt[3]{7.7}$.

$$L(x) = f(a) + f'(a)(x - a)\,; \quad a = 8$$

$$f(x) = x^{\frac{1}{3}} \qquad \Longrightarrow \qquad f(8) = \sqrt[3]{8} = 2$$

$$f'(x) = \tfrac{1}{3}x^{-\frac{2}{3}} \quad \Longrightarrow \qquad f'(8) = \frac{1}{3}\left(\frac{1}{\left(\sqrt[3]{8}\right)^2}\right) = \frac{1}{3}\left(\frac{1}{2^2}\right) = \frac{1}{12}$$

$$L(x) = 2 + \tfrac{1}{12}(x - 8)$$

$$\sqrt[3]{8.5} \approx L(8.5) = 2 + \tfrac{1}{12}(8.5 - 8) \approx 2.0417$$

$$\sqrt[3]{7.7} \approx L(7.7) = 2 + \tfrac{1}{12}(7.7 - 8) \approx 1.975.$$

Ex. Approximate $\sin(3°)$ with a linear approximation.

Since for the function $f(x) = sinx$, x is a real number (i.e., radians) and not degrees, we first need to convert $3°$ into radians.

$$\frac{d}{360} = \frac{R}{2\pi}\,; \qquad R = \frac{2\pi d}{360} = \frac{2\pi(3)}{360} = \frac{\pi}{60}\,.$$

$\frac{\pi}{60}$ is pretty close to $x = 0$, so let's find the linear approximation around $a = 0$.

$$L(x) = f(a) + f'(a)(x - a)\,; \quad a = 0$$

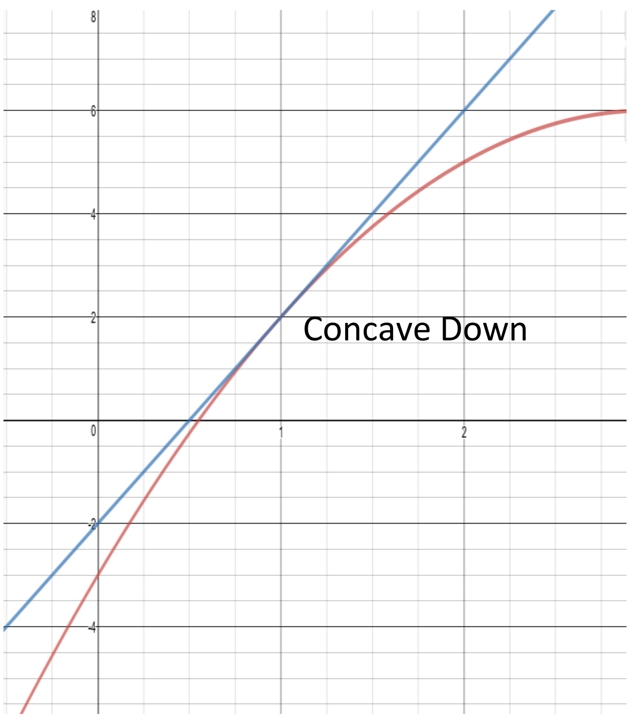
$$f(x) = sinx \qquad \Longrightarrow \qquad f(0) = sin0 = 0$$

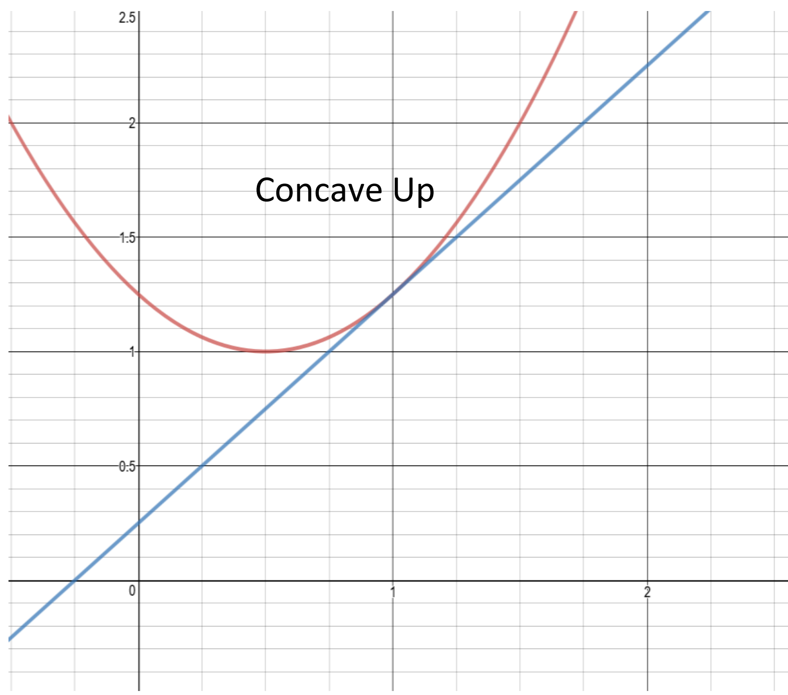
$$f'(x) = cosx \qquad \Longrightarrow \qquad f'(0) = cos0 = 1$$

$$L(x) = 0 + 1(x - 0) = x$$

$$\sin(3°) \approx L\left(\frac{\pi}{60}\right) = \frac{\pi}{60}.$$

Notice that the linear approximation will be too high if the graph of $y = f(x)$ is concave down between a and the point being approximated and too low if the graph of $y = f(x)$ is concave up between a and the point being approximated.





Ex. Is the approximation for $\sin(3°)$ too high or too low?

$f(x) = sinx$

$f'(x) = cosx$

$f''(x) = -sinx$

$a = 0$ and the point being approximated is $\frac{\pi}{60}$ and $0 < \frac{\pi}{60} < \frac{\pi}{2}$.

Since $sinx$ is positive for $0 < x < \frac{\pi}{2}$, $f''(x) = -sinx < 0$ for $0 < x < \frac{\pi}{60}$.

Thus the graph of $f(x) = sinx$ is concave down on $0 < x < \frac{\pi}{60}$, so the approximation is too high.

We can also adapt our linear approximation function to allow us to approximate how much a function **changes** over an interval.

$f(x) \approx L(x) = f(a) + f'(a)(x - a)$ for all x in I.

So $f(x) - f(a) \approx f'(a)(x - a)$

or $\Delta \boldsymbol{y} \approx \boldsymbol{f'(a)}\Delta \boldsymbol{x}$.

In other words, the value of a function $f(x)$ changes by approximately $f'(a)\Delta x$ when x changes by $\Delta x = x - a$.

Ex. Approximate the change in the area of a circle when the radius goes from 3 inches to 3.1 inches.

$A(r) = \pi r^2$

$\Delta A \approx A'(a)\Delta r\,; \quad a = 3.$

$\Delta r = 3.1 - 3 = 0.1$

$A'(r) = 2\pi r$

$A'(a) = 2\pi(3) = 6\pi$

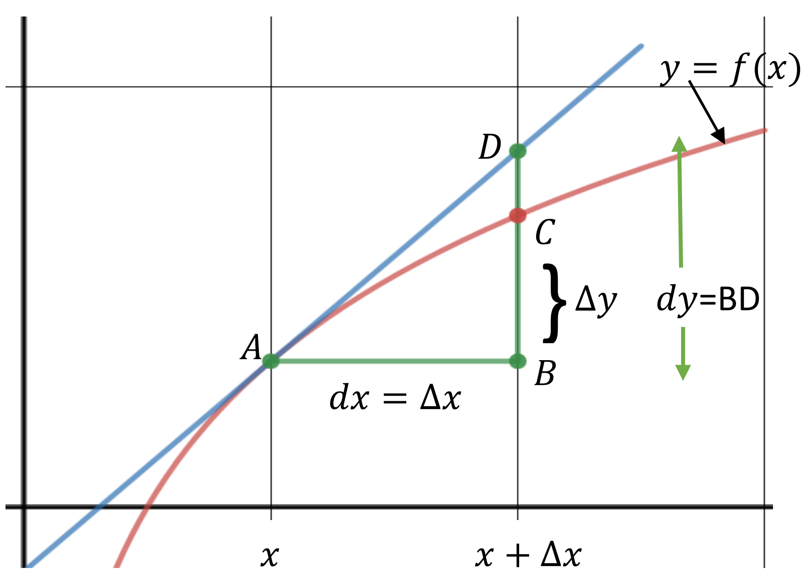
$\Delta A \approx A'(a)\Delta r = 6\pi(0.1) = 0.6\pi.$

<u>Differentials</u>

Def. Let f be differentiable on an interval containing x. A small change in x, Δx, is denoted by dx. The corresponding change in f is approximated by the differential $\boldsymbol{dy = f'(x)dx}$.

That is: $\Delta y = f(x + dx) - f(x) \approx dy = f'(x)dx.$

So dy approximates the change in the function $f(x)$ from x to $x + \Delta x$.



$\Delta y = f(x + dx) - f(x)$

$\Delta y \approx dy = f'(x)dx$

Ex. Let $f(x) = 3sin^2x$. Calculate dy. Approximate the change in the value of this function from $x = \frac{\pi}{6}$ to $x = \frac{\pi}{6} + .2$.

$$dy = f'(x)dx$$

$$f'(x) = 3(2)sinx(cosx) = 6sinx(cosx)$$

at $x = \frac{\pi}{6}$,

$$f'\left(\frac{\pi}{6}\right) = 6(sin\frac{\pi}{6})\left(cos\frac{\pi}{6}\right) = 6\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}$$

$$dx = \Delta x = 0.2$$

$$\Delta y = f(x + dx) - f(x) \approx dy = f'(x)dx = \frac{3\sqrt{3}}{2}\left(\frac{1}{5}\right) = \frac{3\sqrt{3}}{10}.$$

Ex. The radius of a sphere is measured to be 21cm with an error of at most 0.05cm. What is the maximum error in the Volume? What is the maximum percentage error? What is the maximum percentage error in the radius?

$$V = \frac{4}{3}\pi r^3$$

$$\Delta V \approx dV = V'(r)dr$$

$$V'(r) = 4\pi r^2$$

$$r = 21, \;\; dr = \Delta r = .05$$

maximum error in $V = \Delta V \approx (V'(21))(0.05) = 4\pi(21)^2(0.05) \approx 277cm^3$.

maximum % error in $V \approx \frac{dV}{V} = \frac{4\pi(21)^2(0.05)}{\frac{4}{3}\pi(21)^3} \approx 0.007 = 0.7\%$.

maximum % error in $r \approx \frac{dr}{r} = \frac{0.05}{21} \approx 0.002 = 0.2\%$.

Ex. The edge of a cube is measured to be 5 in. with a possible error of 0.01 in. Approximate the maximum error in the volume and the maximum percentage error in the volume.

$V = s^3\,; \quad s = 5, \quad ds = 0.01.$

$\Delta V \approx dV = V'(s)ds = 3s^2 ds.$

maximum error in $V = \Delta V \approx 3(5)^2(0.01) = 0.75in^3.$

maximum % error in $V \approx \frac{dV}{V} = \frac{0.75}{5^3} = \frac{0.75}{125} = 0.006 = 0.6\%.$