Optimization Problems

In Optimization problems we are searching for a global maximum or minimum of a function.

Steps to solving an Optimization problem

1. Read the problem **carefully** (read it 2 or 3 times if necessary). Identify what quantity you are being asked to maximize or minimize. What are the unknowns in the problem and what information are you given about them?

2. Draw a picture (if possible)

3. Assign letters to the unknown quantities and the thing you are trying to maximize or minimize, Q (**Q is called the Objective Function**).

4. Express Q in terms of the quantities in the problem.

5. If necessary, find relationships in the problem that allow you to write Q in terms of a single unknown. Write down the domain of Q.

6. Find the absolute maximum or minimum of Q.

7. Make sure you have answered the question raised in the problem.

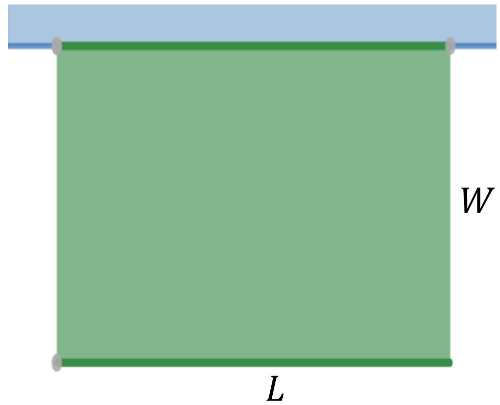
Recall the following theorem:

Theorem (This is very useful!!) If $f(x)$ is a continuous function and c is the only critical point for $f(x)$ on an interval (open, closed, bounded or unbounded):

a. If c is a relative minimum then, c is an absolute minimum on that interval.

b. If c is a relative maximum then, c is an absolute maximum on that interval.

Ex. A farmer has 2400 ft. of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?



$2W + L = 2400$

$A = LW$ (A is the objective function).

We need A to be a function of 1 variable

not 2 (L, W).

$2W + L = 2400 \quad \Longrightarrow \quad L = 2400 - 2W$

$A = LW = W(2400 - 2W); \qquad 0 \leq W \leq 1200,$ since $0 \leq L$.

We want to find the global maximum of A subject to $0 \leq W \leq 1200$.

$A = W(2400 - 2W) = 2400W - 2W^2$

$A'(W) = 2400 - 4W = 0 \implies W = 600.$

Determine whether $W = 600$ is a local maximum/minimum or neither.

By the second derivative test:

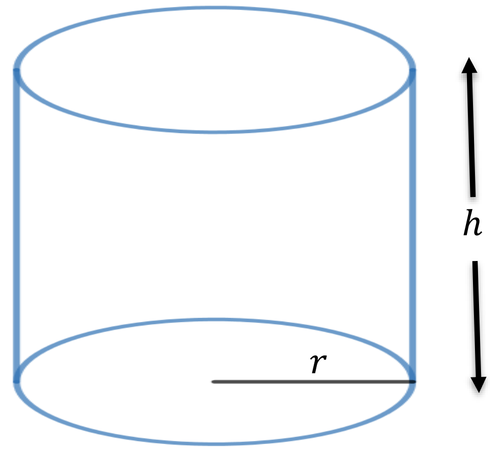
$A''(W) = -4 \implies A''(600) = -4 < 0 \implies W = 600$ a local max.

Since $W = 600$ is the only critical point on $0 \leq W \leq 1200$, it's a global max.

$W = 600 \implies L = 2400 - 2(600) = 1200.$

Dimensions of largest area: $W = 600ft, \quad L = 1200ft.$

Ex. A cylindrical can is to be made with a volume of 1000 cu. cm. Find the dimensions of the can that will minimize the cost of the metal to manufacture the can (i.e., minimize the surface area).



$V = \pi r^2 h = 1000$

$SA = 2\pi rh + 2\pi r^2$

We want to minimize SA subject to $V = 1000$. But the formula for SA has 2 unknowns (r, h). To turn SA into a function of 1 variable we use:

$$\pi r^2 h = 1000 \quad \Longrightarrow \quad h = \frac{1000}{\pi r^2}$$

$$SA = 2\pi rh + 2\pi r^2 = 2\pi r\left(\frac{1000}{\pi r^2}\right) + 2\pi r^2$$

$$SA(r) = \frac{2000}{r} + 2\pi r^2; \qquad \text{subject to } 0 < r.$$

Now let's find a global minimum for $SA(r)$.

$$SA'(r) = -\frac{2000}{r^2} + 4\pi r = 0$$

$$\frac{-2000+4\pi r^3}{r^2} = 0 \quad \Longrightarrow \quad -2000 + 4\pi r^3 = 0$$

$$4\pi r^3 = 2000 \qquad \Longrightarrow \qquad r^3 = \frac{500}{\pi} \quad \Longrightarrow \quad r = \sqrt[3]{\frac{500}{\pi}}.$$

Determine if $r = \sqrt[3]{\frac{500}{\pi}}$ is a local maximum/minimum or neither.

$SA''(r) = \frac{4000}{r^3} + 4\pi > 0$ for all $0 < r,$ so $SA''\left(\sqrt[3]{\frac{500}{\pi}}\right) > 0.$

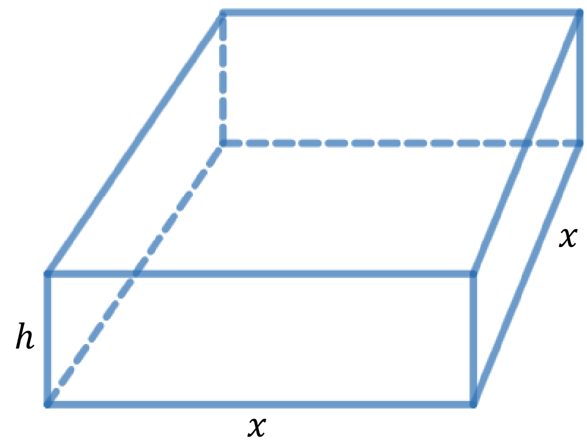
$\Longrightarrow r = \sqrt[3]{\frac{500}{\pi}}$ is a local minimum by the 2nd derivative test.

Since there is only one critical point on the interval $0 < r,\ r = \sqrt[3]{\frac{500}{\pi}}$ is a global minimum.

$$h = \frac{1000}{\pi r^2} \Longrightarrow h = \frac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2} = \frac{1000}{(\pi)^{\frac{1}{3}}(500)^{\frac{2}{3}}} = \frac{2(500)}{(\pi)^{\frac{1}{3}}(500)^{\frac{2}{3}}} = 2\left(\frac{500}{\pi}\right)^{\frac{1}{3}}.$$

Dimensions that minimize cost: $r = \sqrt[3]{\frac{500}{\pi}}\, cm$, $\qquad h = 2\left(\frac{500}{\pi}\right)^{\frac{1}{3}} cm.$

Ex. An open top rectangular box with a square base is to be constructed to hold 1000 cu. ft. The material for the base costs \$10/sq. ft. and the material for the sides costs \$5/sq. ft. Find the dimensions that minimize the cost of making the box.



Cost of base= $10x^2$

Cost of sides= $4xh(5) = 20xh$

Total cost= $10x^2 + 20xh$

$V = x^2h = 1000$

$TC = 10x^2 + 20xh;$ but this is a function of 2 variables. So use:

$$x^2h = 1000 \quad \Longrightarrow \quad h = \frac{1000}{x^2}.$$

$$TC = 10x^2 + 20xh = 10x^2 + 20x\left(\frac{1000}{x^2}\right)$$

$$TC = 10x^2 + \frac{20000}{x}; \qquad 0 < x.$$

Now find the global minimum of TC.

$$TC'(x) = 20x - \frac{20000}{x^2} = 0$$

$$\frac{20x^3 - 20000}{x^2} = 0$$

$$20x^3 - 20000 = 0$$

$$20x^3 = 20000$$

$$x^3 = 1000 \quad \Longrightarrow \quad x = 10.$$

Determine if $x = 10$ is a local maximum/minimum or neither.

$TC''(x) = 20 + \frac{40000}{x^3} > 0$ for all $0 < x$. Thus $TC''(10) > 0$

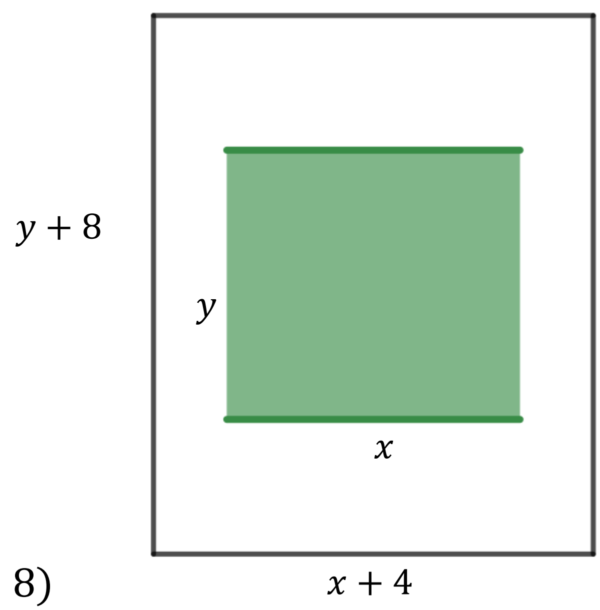
and $x = 10$ is a local minimum.

Since $x = 10$ was the only critical point, $x = 10$ is a global minimum.

$$h = \frac{1000}{x^2} \implies h = \frac{1000}{(10)^2} = 10.$$

Dimensions that minimize cost: $10ft \times 10ft \times 10ft.$

Ex. A rectangular shaped garden needs to be 32 sq. ft. A path is built around the garden so that it is 4ft. wide along the top and bottom and 2 ft. wide along the sides. What should the dimensions of the garden be to minimize the area used by the garden and the paths together?



$xy = 32$

We want to minimize:

$A = (x + 4)(y + 8).$

A has 2 unknowns so use:

$$xy = 32 \implies y = \frac{32}{x}$$

$$A = (x + 4)(y + 8) = (x + 4)(\frac{32}{x} + 8)$$

$$A(x) = 32 + 8x + \frac{128}{x} + 32$$

$A(x) = 8x + \frac{128}{x} + 64; \qquad 0 < x.$

$A'(x) = 8 - \frac{128}{x^2} = 0$

$\frac{8x^2 - 128}{x^2} = 0$

$8x^2 - 128 = 0$

$x^2 - 16 = 0 \quad \Longrightarrow \quad x = \pm 4.$ But $0 < x$, so $x = 4.$

Determine if $x = 4$ is a local maximum/minimum or neither.

$A''(x) = \frac{256}{x^3} > 0$ for all $x > 0$. So $A''(4) > 0$, and $x = 4$ is a local min.

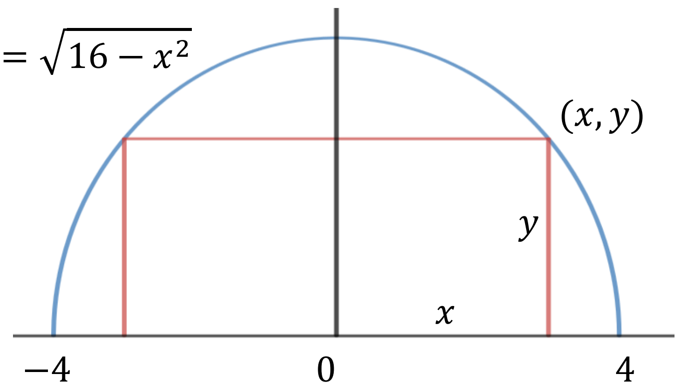
Since $x = 4$ is the only critical point on $x > 0$, $x = 4$ is a global min.

$y = \frac{32}{x} \Longrightarrow y = \frac{32}{4} = 8.$

Dimensions of garden for minimum area: $x = 4ft, \quad y = 8ft.$

Ex. Find the area of the largest rectangle that can be inscribed in the semi-circle

$$y = \sqrt{16 - x^2}.$$



$A = 2xy$; where $y = \sqrt{16 - x^2}$.

To make A a function of 1 variable:

$$A = 2xy = 2x\sqrt{16 - x^2}$$

$$A = 2x(16 - x^2)^{\frac{1}{2}}; \qquad 0 \le x \le 4.$$

Now find the global maximum of A.

$$A'(x) = 2[x\left(\tfrac{1}{2}\right)(16 - x^2)^{-\frac{1}{2}}(-2x) + (16 - x^2)^{\frac{1}{2}}]$$

$$= 2\left[-\frac{x^2}{\sqrt{16-x^2}} + \sqrt{16 - x^2}\right] = 0$$

$$\frac{-x^2+(16-x^2)}{\sqrt{16-x^2}} = 0$$

$$-x^2 + (16 - x^2) = 0$$

$$16 = 2x^2 \quad \Longrightarrow \quad x^2 = 8 \quad \Longrightarrow \quad x = \pm\sqrt{8}.$$

But $0 \le x \le 4$ so $x = \sqrt{8}$.

Determine if $x = \sqrt{8}$ is a local maximum/minimum or neither.

It's messy to calculate $A''(x)$, but this is a closed interval so we just need to check the values of A at the critical points and the endpoints.

$$A(0) = 2(0)(\sqrt{16}) = 0$$

$$A(\sqrt{8}) = 2(\sqrt{8})(\sqrt{16-8}) = 16$$

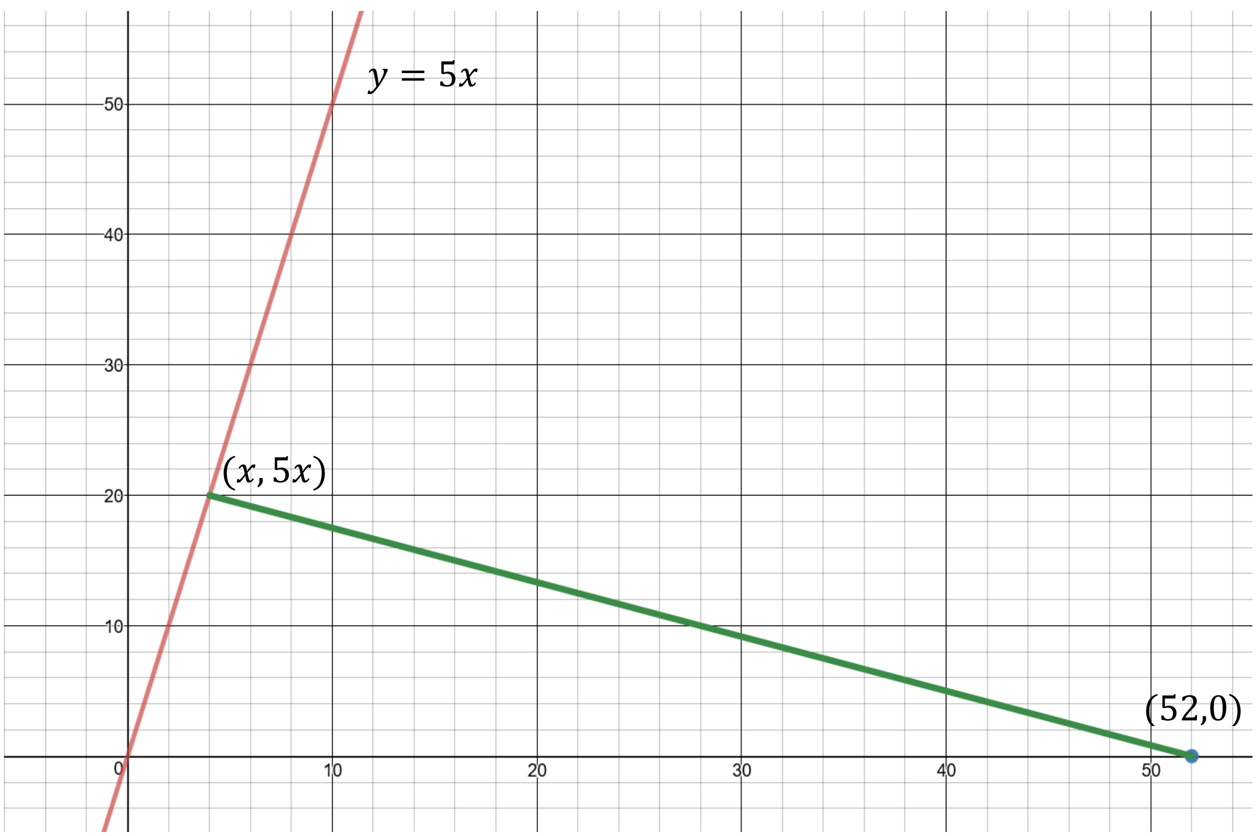
$$A(4) = 2(4)(\sqrt{0}) = 0.$$

So the absolute maximum of A occurs at $x = \sqrt{8}$.

Largest area: $A(\sqrt{8}) = 16$ sq. units.

Ex. Find the closest point on the line $y = 5x$ to $(52,0)$.

Note: It's easier to find the point where the square of the distance is a minimum.



If $(x, y) = (x, 5x)$ is any point on the line $y = 5x$, then the distance from that point to $(52,0)$ is:

$$D = \sqrt{(x-52)^2 + (y-0)^2} = \sqrt{(x-52)^2 + (5x-0)^2}\ .$$

This distance function will have an absolute

minimum at the same point as D^2, which

is easier to work with algebraically.

$$D^2 = (x-52)^2 + 25x^2; \quad x \in \mathbb{R}.$$

$$\frac{d}{dx}\left(D^2\right) = 2(x-52) + 50x$$

$$= 52x - 104 = 0 \quad \Longrightarrow \quad x = 2.$$

Determine if $x = 2$ is a local maximum/minimum or neither.

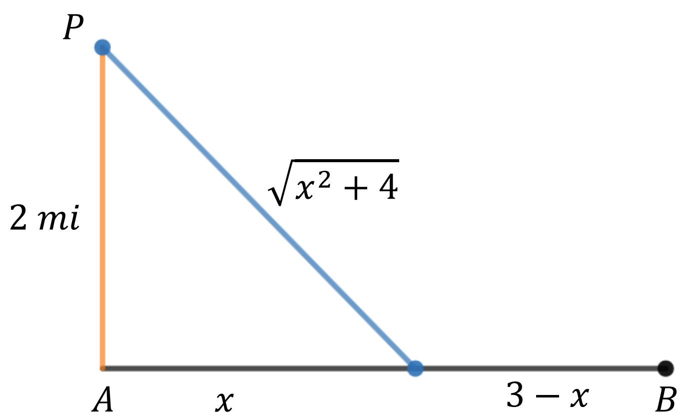
$\frac{d^2}{dx^2}\left(D^2\right) = 52$ for all $x,$ so at $x = 2$ it is also positive.

So $x = 2$ is a local minimum.

Since it's also the only critical point it's a global minimum.

Closest point on $y = 5x$ to $(52,0)$: $(2,10)$.

Ex. A woman at point P is in a boat 2 miles from the nearest point on the straight coast (point A). She is to go to a point B, located 3 miles down the coast. If she can row at 2 miles per hour and walk at 3 miles per hour, toward what point on the coast should she row in order to reach point B in the least time?



walks at 3mph

rows at 2mph

$0 \le x \le 3.$

Time=distance/rate

$$T(x) = \frac{\sqrt{x^2+4}}{2} + \frac{3-x}{3} = \frac{1}{2}(x^2+4)^{\frac{1}{2}} + \frac{1}{3}(3-x) \ ; \qquad 0 \le x \le 3$$

$$T'(x) = \frac{1}{4}(x^2+4)^{-\frac{1}{2}}(2x) - \frac{1}{3}$$

$$= \frac{x}{2\sqrt{x^2+4}} - \frac{1}{3} = 0.$$

$$\frac{x}{2\sqrt{x^2+4}} = \frac{1}{3}$$ now cross multiply

$$3x = 2\sqrt{x^2+4}$$ square both sides

$$9x^2 = 4(x^2+4)$$

$$5x^2 = 16 \implies x^2 = \frac{16}{5}$$

$$x = \pm\frac{4}{\sqrt{5}}; \quad \text{but } \ 0 \le x \le 3, \text{ so } x = \frac{4}{\sqrt{5}}.$$

Determine if $x = \frac{4}{\sqrt{5}}$ is a local maximum/minimum or neither.

One can calculate $T''(x)$, or since this is a closed interval we just need to check the value of $T(x)$ at the critical point(s) and the endpoints.

$$T(0) = \frac{\sqrt{4}}{2} + 1 = 3$$

$$T\left(\frac{4}{\sqrt{5}}\right) = \frac{\sqrt{\frac{16}{5}+4}}{2} + \frac{3-\frac{4}{\sqrt{5}}}{3} \approx 1.745$$

$$T(3) = \frac{\sqrt{9+4}}{2} + 0 \approx 4.803$$

So to minimize the time she should row to the point $3 - \frac{4}{\sqrt{5}}$ miles west of B.