# Derivatives and the Shapes of Graphs

## The Significance of the First Derivative

Increasing/Decreasing Test

a. If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that interval

b. If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on that interval.

**To find where $f'(x) > 0$ or where $f'(x) < 0$ we first want to find where $f'(x) = 0$ or where $f'(x)$ is undefined. We then "test" the sign of $f'(x)$ at points in between the points where $f'(x) = 0$ or where $f'(x)$ is undefined.**

Ex. Find where the function $f(x) = x^3 - 3x^2 - 9x + 2$ is increasing and where it is decreasing.

First find where $f'(x) = 0$.

$$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3)$$

$$= 3(x-3)(x+1) = 0 \implies x = 3, -1.$$

So $f'(x) = 0$ when $x = 3, -1$.

Next test the sign of $f'(x)$ for a single point in each of the intervals: $x < -1$, $-1 < x < 3$, and $3 < x$. $f'(x)$ will have the same sign for every point in the interval.

To test the sign of $f'(x)$ for $x < -1$, choose any point in that interval, for example $x = -2$, and find the sign of $f'(x)$.

$$f'(-2) = 3(-2-3)(-2+1) = 3(-5)(-1) = 15 > 0.$$

So $f'(x) > 0$ for every point in $x < -1$.

To test the sign of $f'(x)$ for $-1 < x < 3$, choose any point in that interval, for example $x = 0$, and find the sign of $f'(x)$.

$$f'(0) = 3(0-3)(0+1) = 3(-3)(1) = -9 < 0.$$

So $f'(x) < 0$ for every point in $-1 < x < 3$.

To test the sign of $f'(x)$ for $3 < x$, choose any point in that interval, for example $x = 4$, and find the sign of $f'(x)$.

$$f'(4) = 3(4-3)(4+1) = 3(1)(5) = 15 > 0.$$

So $f'(x) > 0$ for every point in $3 < x$.

sign of $f'(x)$ ______+______|________−________|____+____

$-1$ $3$

So $f(x)$ is increasing when $x < -1$ or $3 < x$.

$f(x)$ is decreasing when $-1 < x < 3$.

Ex. Find where the function $f(x) = x^3 - 3x^2 + 4$ is increasing and where it is decreasing.

First find where $f'(x) = 0$.

$f'(x) = 3x^2 - 6x = 3x(x-2) = 0 \quad \Longrightarrow \quad x = 0, 2.$

So $f'(x) = 0$ when $x = 0, 2$.

Next test the sign of $f'(x)$ for a single point in each of the intervals: $x < 0$, $0 < x < 2$, and $2 < x$. $f'(x)$ will have the same sign for every point in the interval.

To test the sign of $f'(x)$ for $x < 0$, choose any point in that interval, for example $x = -1$, and find the sign of $f'(x)$.

$$f'(-1) = 3(-1)(-1-2) = 9 > 0.$$

So $f'(x) > 0$ for every point in $x < 0$.

To test the sign of $f'(x)$ for $0 < x < 2$, choose any point in that interval, for example $x = 1$, and find the sign of $f'(x)$.

$$f'(1) = 3(1)(1-2) = -3 < 0.$$

So $f'(x) < 0$ for every point in $0 < x < 2$.

To test the sign of $f'(x)$ for $2 < x$, choose any point in that interval, for example $x = 3$, and find the sign of $f'(x)$.

$$f'(3) = 3(3)(3-2) = 9 > 0.$$

So $f'(x) > 0$ for every point in $2 < x$.

sign of $f'(x)$ ______+______|______$-$______|____+____

0 2

So $f(x)$ is increasing when $x < 0$ or $2 < x$.

$f(x)$ is decreasing when $0 < x < 2$.

Ex. Suppose $f'(x) = \frac{6(x^2-1)}{(x^2-9)^2}$ . Find where $f(x)$ is increasing and where it's decreasing.

Factor $f'(x)$ completely to determine where $f'(x) = 0$ or is undefined.

$$f'(x) = \frac{6(x^2-1)}{(x^2-9)^2} = \frac{6(x-1)(x+1)}{(x-3)^2(x+3)^2}$$

So $f'(x) = 0$ when $x = \pm 1$, and $f'(x)$ is undefined when $x = \pm 3$.

So we need to test the sign of $f'(x)$ on the following intervals:

$$x < -3, \quad -3 < x < -1, \quad -1 < x < 1, \quad 1 < x < 3, \quad 3 < x.$$

So we choose a point in each interval and test the sign of $f'(x)$. Remember, we only care about the sign of the derivative:

$$f'(-4) = \frac{6(-4-1)(-4+1)}{(-4-3)^2(-4+3)^2} = \frac{6(-)(-)}{(-)^2(-)^2} = \frac{+}{+} = +$$

$$f'(-2) = \frac{6(-2-1)(-2+1)}{(-2-3)^2(-2+3)^2} = \frac{6(-)(-)}{(-)^2(+)^2} = \frac{+}{+} = +$$

$$f'(0) = \frac{6(0-1)(0+1)}{(0-3)^2(0+3)^2} = \frac{6(-)(+)}{(-)^2(+)^2} = \frac{-}{+} = -$$

$$f'(2) = \frac{6(2-1)(2+1)}{(2-3)^2(2+3)^2} = \frac{6(+)(+)}{(-)^2(+)^2} = \frac{+}{+} = +$$

$$f'(4) = \frac{6(4-1)(4+1)}{(4-3)^2(4+3)^2} = \frac{6(+)(+)}{(-)^2(+)^2} = \frac{+}{+} = +$$

| sign of $f'(x)$ | + | \| | + | \| | − | \| | + | \| | + |
|---|---|---|---|---|---|---|---|---|---|
| | | $-3$ | | $-1$ | | $1$ | | $3$ | |

$f(x)$ is increasing when $x < -3$ or $-3 < x < -1$ or $1 < x < 3$ or $3 < x$.

$f(x)$ is decreasing when $-1 < x < 1$.

Note: $f(x)$ is not increasing on $x < -1$ because $f'(x)$ is infinite at $x = -3$. If $f'(-3)$ was equal to 0 then we could have said that $f(x)$ is increasing on $x < -1$ .

A similar comment applies to $f(x)$ for $x > 1$.

Ex. Suppose $f'(x) = \frac{3(4-x^2)}{x^2-25}$. Find where $f(x)$ is increasing where it's decreasing.

Factor $f'(x)$ completely to determine where $f'(x) = 0$ or is undefined.

$$f'(x) = \frac{3(4-x^2)}{x^2-25} = \frac{3(2-x)(2+x)}{(x-5)(x+5)}$$

So $f'(x) = 0$ when $x = \pm 2$, and $f'(x)$ is undefined when $x = \pm 5$.

So we need to test the sign of $f'(x)$ on the following intervals:

$$x < -5, \quad -5 < x < -2, \quad -2 < x < 2, \quad 2 < x < 5, \quad 5 < x.$$

So we choose a point in each interval and test the sign of $f'(x)$. Remember, we only care about the sign of the derivative:

$$f'(-6) = \frac{3\big(2-(-6)\big)(2+(-6))}{(-6-5)(-6+5)} = \frac{6(+)(-)}{(-)(-)} = \frac{-}{+} = -$$

$$f'(-4) = \frac{3\big(2-(-4)\big)(2+(-4))}{(-4-5)(-4+5)} = \frac{6(+)(-)}{(-)(+)} = \frac{-}{-} = +$$

$$f'(0) = \frac{3\big(2-(0)\big)(2+(0))}{(0-5)(0+5)} = \frac{6(+)(+)}{(-)(+)} = \frac{+}{-} = -$$

$$f'(4) = \frac{3\big(2-(4)\big)(2+(4))}{(4-5)(4+5)} = \frac{6(-)(+)}{(-)(+)} = \frac{-}{-} = +$$

$$f'(6) = \frac{3\big(2-(6)\big)(2+(6))}{(6-5)(6+5)} = \frac{6(-)(+)}{(+)(+)} = \frac{-}{+} = - .$$

sign of $f'(x)$ — | + | — | + | —

$-5 \qquad -2 \qquad 2 \qquad 5$

$f(x)$ is increasing when $-5 < x < -2$ or $2 < x < 5$.

$f(x)$ is decreasing when $x < -5$ or $-2 < x < 2$ or $5 < x$.

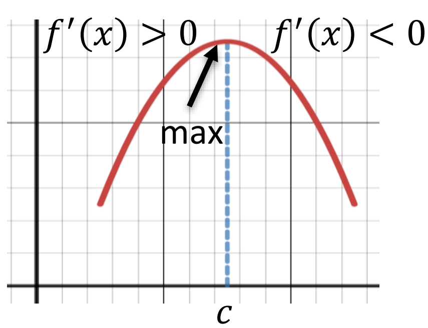
**First Derivative Test**: Suppose $c$ is a critical point of a continuous function $f(x)$.

a. If $f'(x)$ changes from positive to negative at $x = c$, then $x = c$ is a local maximum.

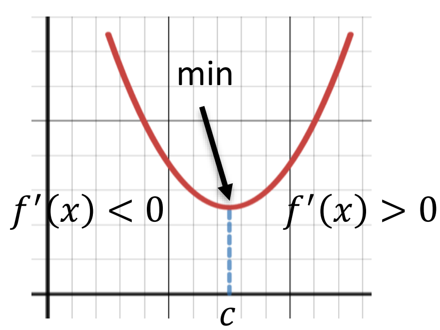
b. If $f'(x)$ changes from negative to positive at $x = c$, then $x = c$ is a local minimum.

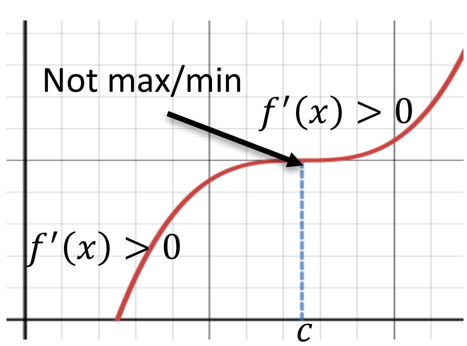
c. If $f'(x)$ doesn't change sign at $x = c$ (ie., it stays positive or stays negative) then $f(x)$ does not have a local max or min at $x = c$.

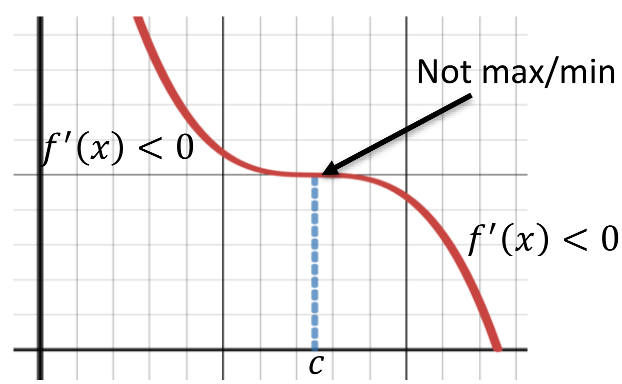
$a.$




$b.$



Ex. Find all relative/local maxima and minima for $f(x) = x^3 - 3x^2 - 9x + 2.$

By the first derivative test we need to find all critical points and observe the sign of the first derivative as we pass through those points.

$$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1) = 0$$

So $f'(x) = 0$ when $x = 3, -1.$ Thus $x = 3, -1$ are the critical points.

To determine the sign of $f'(x)$ we need to test its sign on the intervals:

$$x < -1, \quad -1 < x < 3, \quad 3 < x.$$

We did this earlier and found:

sign of $f'(x)$ ____+____|_____−_____|____+____ .

$-1$ $3$

So by the first derivative test:

$$x = -1, \;\; f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 2 = 7, \quad (-1, 7)$$

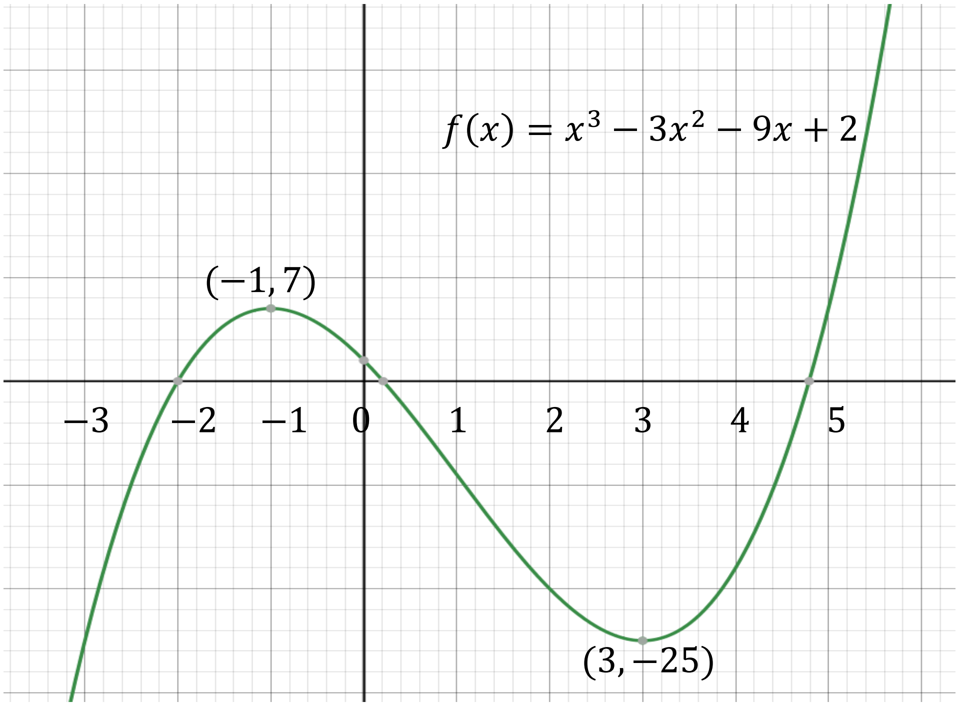
is a relative maximum since the derivative is going from positive to negative as

$x$ increases through $x = -1.$

$$x = 3, \;\; f(3) = (3)^3 - 3(3)^2 - 9(3) + 2 = -25, \quad (3, -25)$$

is a relative minimum since the derivative is going from negative to positive as

$x$ increases through $x = 3.$




Ex. Find all relative/local maxima and minima for $f(x) = x^3 - 3x^2 + 4$.

By the first derivative test we need to find all critical points and observe the sign of the first derivative as we pass through those points.

$$f'(x) = 3x^2 - 6x = 3(x)(x-2) = 0$$

So $f'(x) = 0$ when $x = 0, 2$. Thus $x = 0, 2$ are the critical points.

To determine the sign of $f'(x)$ we need to test its sign on the intervals:

$$x < 0, \quad 0 < x < 2, \qquad 2 < x.$$

We did this earlier and found:

sign of $f'(x)$ ______+______|________—________|____+____ .

$0$ $2$

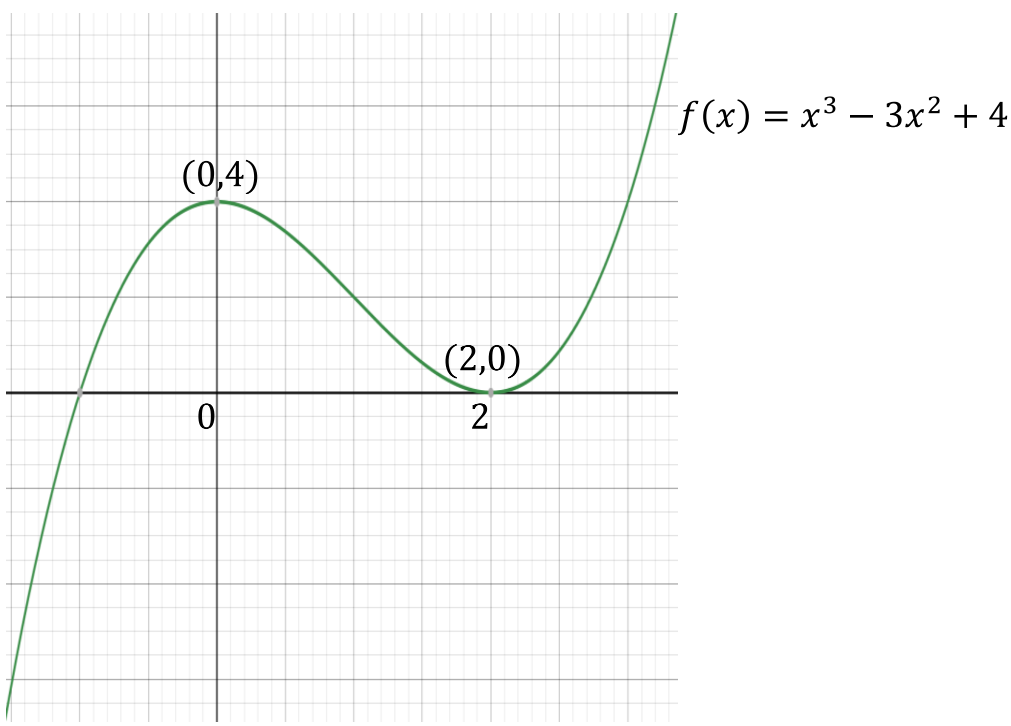
So by the first derivative test:

$x = 0, \; f(0) = (0)^3 - 3(0)^2 + 4 = 2, \qquad (0,4)$

is a relative maximum since the derivative is going from positive to negative as $x$ increases through $x = 0$.

$x = 2, \; f(2) = (2)^3 - 3(2)^2 + 4 = 0, \qquad (2,0)$

is a relative minimum since the derivative is going from negative to positive as $x$ increase through $x = 2$.



Ex. Find the $x$ coordinate of any relative maxima/minima of $f(x)$ if

$f'(x) = \frac{6(x^2-1)}{(x^2-9)^2}$. Assume that the domain of $f(x)$ is all real numbers

except $x = 3, -3$.

In an earlier example we found the sign of $f'(x) = \frac{6(x^2-1)}{(x^2-9)^2}$ to be:

| sign of $f'(x)$ | $+$ | | $+$ | | $-$ | | $+$ | | $+$ |
|---|---|---|---|---|---|---|---|---|---|
| | | $-3$ | | $-1$ | | $1$ | | $3$ | |

So by the first derivative test since $f(x)$ is continuous at $x = -1, 1$:

$x = -1$ is a relative maximum since $f'(x)$ goes from positive to negative as $x$ goes through $x = -1$.

$x = 1$ is a relative minimum since $f'(x)$ goes from negative to positive as $x$ goes through $x = 1$.

Note: Since we don't know what the function $f(x)$ is we can't say what the $y$ coordinate is for the relative maximum and minimum.

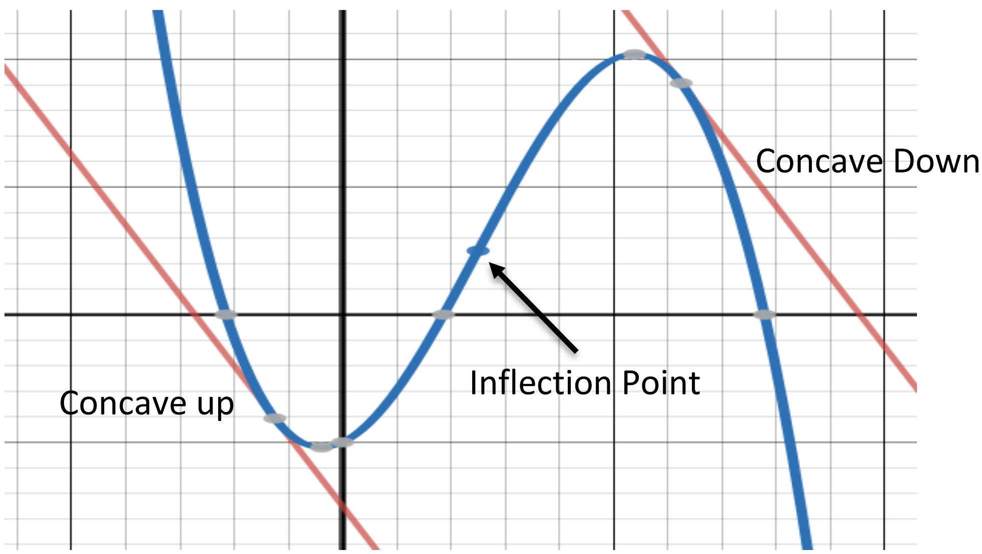
Theorem (This will be useful when we do optimization problems) Suppose $f(x)$ is continuous on an interval $I$ that contains exactly 1 extremum at $x = c$, then

a. If $x = c$ is a local min, then $f(c)$ is the absolute minimum value of $f$ on $I$
b. If $x = c$ is a local max, then $f(c)$ is the absolute maximum value of $f$ on $I$.

<u>The Significance of the Second Derivative</u>

Def. If a graph lies above all of its tangent lines on an interval we call the graph ***Concave Up***. If the graph lies below its tangent lines we call it **Concave Down**.

Def. A point $p$ on a curve $y = f(x)$ is called an **Inflection Point** if $f(x)$ is continuous at $p$ and the curve changes concavity at $p$.




Concavity Test:

a. If $f''(x) > 0$ for all $x$ on an interval then $f(x)$ is concave up on that interval.

b. If $f''(x) < 0$ for all $x$ on an interval then $f(x)$ is concave down on that interval.

Ex. $y = x^4 - 4x^3$

a. Where is $f(x)$ increasing/decreasing?

b. Where does $f(x)$ have local max./min?

c. Where is $f(x)$ concave up/down?

d. Where does $f(x)$ have inflection points?

e. Sketch a graph of $f(x)$.

a. To find where $f(x)$ is increasing/decreasing we have to find the sign of $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 4x^3 - 12x^2 = 4x^2(x-3) = 0 \quad \Longrightarrow \quad x = 0, 3.$$

So we need to test the sign of $\frac{dy}{dx}$ on the intervals:

$$x < 0, \qquad 0 < x < 3, \qquad 3 < x.$$

At $x = -1$, $\quad \frac{dy}{dx} = 4(-1)^2(-1-3) = (+)(-) = -$

At $x = 1$, $\quad \frac{dy}{dx} = 4(1)^2(1-3) = (+)(-) = -$

At $x = 4$, $\quad \frac{dy}{dx} = 4(4)^2(4-3) = (+)(+) = +$

sign of $\frac{dy}{dx}$ ____$-$\_\_\_\_|\_\_\_\_$-$\_\_\_\_|\_\_\_\_$+$____ .

$\qquad\qquad\qquad\qquad 0 \qquad\qquad\qquad 3$

So $f(x)$ is increasing for $3 < x$ and decreasing for $x < 3$.

b. By the first derivative test, $f(x)$ has a relative minimum at:

$$x = 3, \quad y = (3)^4 - 4(3)^3 = -27; \qquad (3, -27).$$

Note: $x = 0$ is not a relative maximum or minimum since $\frac{dy}{dx}$

does not change sign a $x$ goes through that point.

c. To determine the concavity we have to find the sign of $\frac{d^2y}{dx^2}$:

$$\frac{d^2y}{dx^2} = 12x^2 - 24x = 12x(x-2) = 0 \qquad \Longrightarrow \qquad x = 0, 2.$$

So we need to test the sign of $\frac{d^2y}{dx^2}$ on the intervals:

$$x < 0, \qquad 0 < x < 2, \qquad 2 < x.$$

At $x = -1$, $\quad \frac{d^2y}{d^2x} = 12(-1)(-1-2) = (-)(-) = +$

At $x = 1$, $\quad \frac{d^2y}{dx^2} = 12(1)(1-2) = (+)(-) = -$

At $x = 3$, $\quad \frac{d^2y}{dx^2} = 12(3)(3-2) = (+)(+) = +$

sign of $\frac{d^2y}{dx^2}$ ____+____|________−________|____+____ .

0 2

$f(x)$ is concave up when $x < 0$ or $2 < x$.

$f(x)$ is concave down when $0 < x < 2$.

d. $f(x)$ has inflection points at:

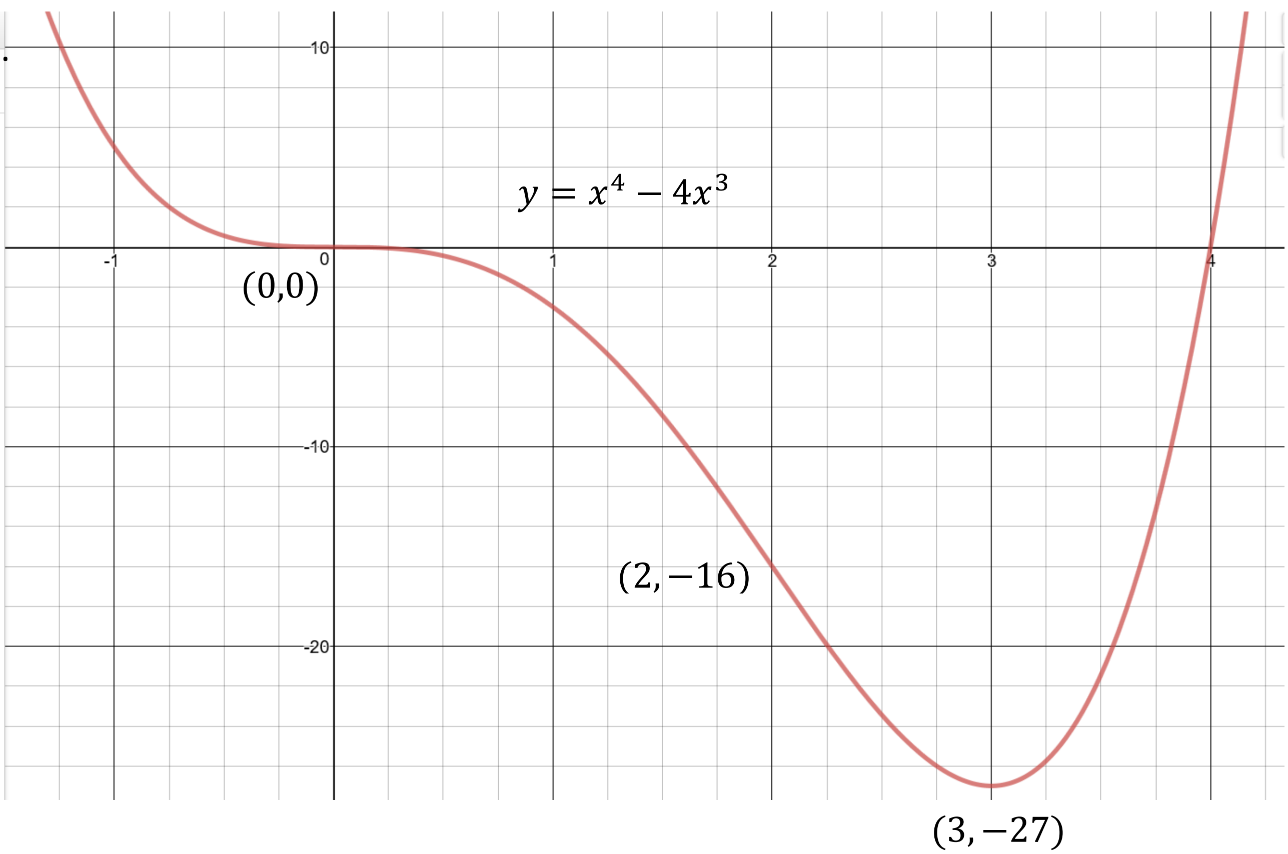
$x = 0, \quad y = (0)^4 - 4(0)^3 = 0; \qquad (0,0)$

since the concavity goes from positive to negative at that point and it's a point of continuity.

$x = 2, \quad y = (2)^4 - 4(2)^3 = -16 \qquad (2, -16)$

since the concavity goes from negative to positive at that point and it's a point of continuity.

e.



Ex. Sketch a graph of $y = f(x)$ with $f'(x) > 0$ for $0 < x < 3$ or $6 < x < 7$

$f'(x) < 0$ for $3 < x < 6$

$f''(x) > 0$ for $0 < x < 1$ or $5 < x < 7$

$f''(x) < 0$ for $1 < x < 5.$

sign of $f'(x)$ ____+____|________−________|____+____

0 3 6 7

$f'(x) > 0$ for $0 < x < 3$ or $6 < x < 7$ $\Longrightarrow$ $f(x)$ is increasing

$f'(x) < 0$ for $3 < x < 6$ $\Longrightarrow$ $f(x)$ is decreasing

By the first derivative test $f(x)$ has a relative maximum at $x = 3$ and a relative minimum at $x = 6.$

sign of $f''(x)$ _____+_____|_____−_____|________+________

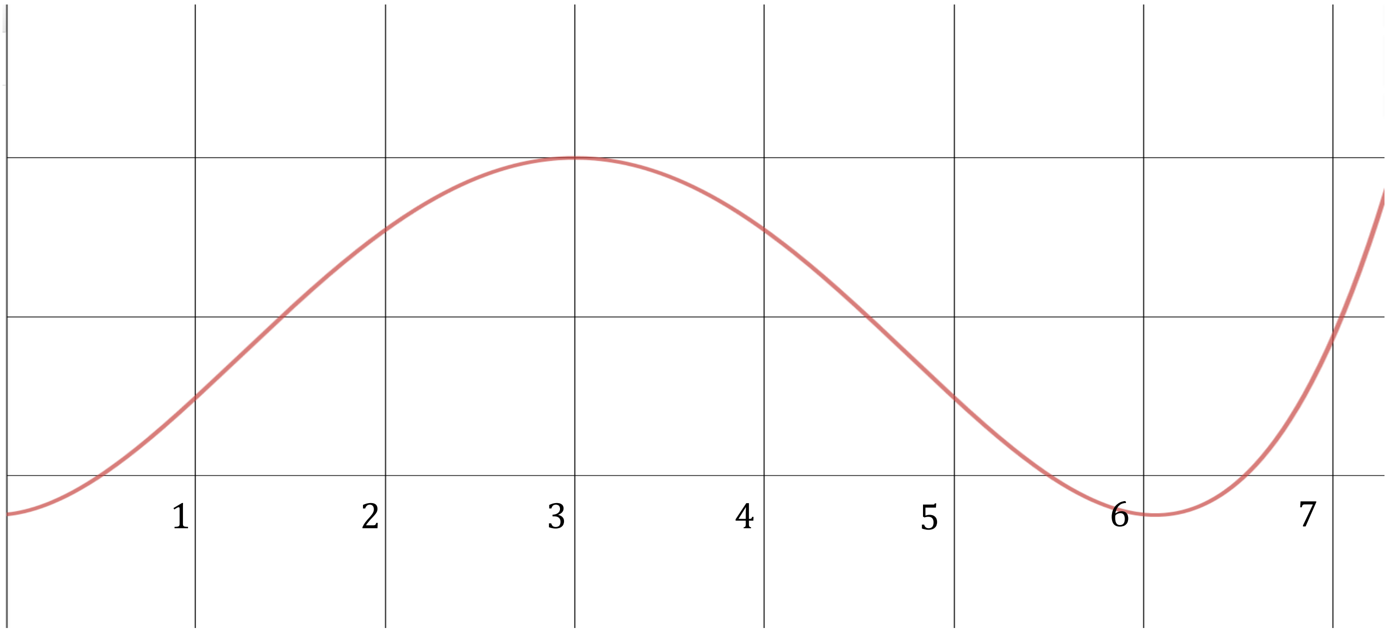
0 1 5 7

$f''(x) > 0$ for $0 < x < 1$ or $5 < x < 7$ $\Longrightarrow$ $f(x)$ is concave up

$f''(x) < 0$ for $1 < x < 5$ $\Longrightarrow$ $f(x)$ is concave down

$f(x)$ has inflection points at $x = 1, 5.$

Note: We can only graph a rough "shape" of $y = f(x)$.

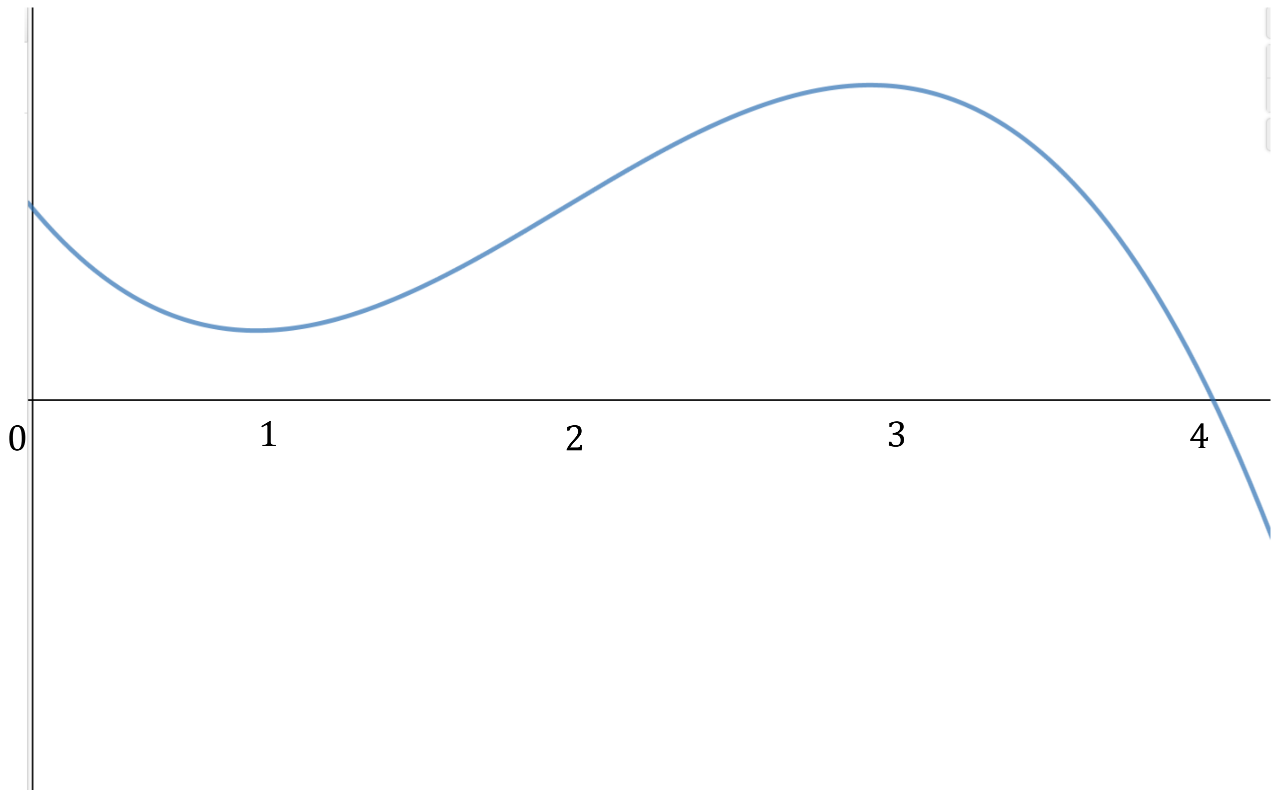



Ex. Sketch a graph of $y = f(x), \quad 0 \le x \le 4$ where:

| $x$ | $f'(x)$ | $f''(x)$ | Incr/Decr | Concave up/down |
|---|---|---|---|---|
| $0 \le x < 1$ | $< 0$ | $> 0$ | Decr | Up |
| $1$ | $0$ | $> 0$ | | Up |
| $1 < x < 2$ | $> 0$ | $> 0$ | Incr | Up |
| $2$ | $> 0$ | $0$ | Incr | |
| $2 < x < 3$ | $> 0$ | $< 0$ | Incr | Down |
| $3$ | $0$ | $< 0$ | | Down |
| $3 < x \le 4$ | $< 0$ | $< 0$ | Decr | Down |

By the first derivative test, $f(x)$ has a local minimum at $x = 1$ and a local maximum at $x = 3$.

$f(x)$ has an inflection point at $x = 2$.

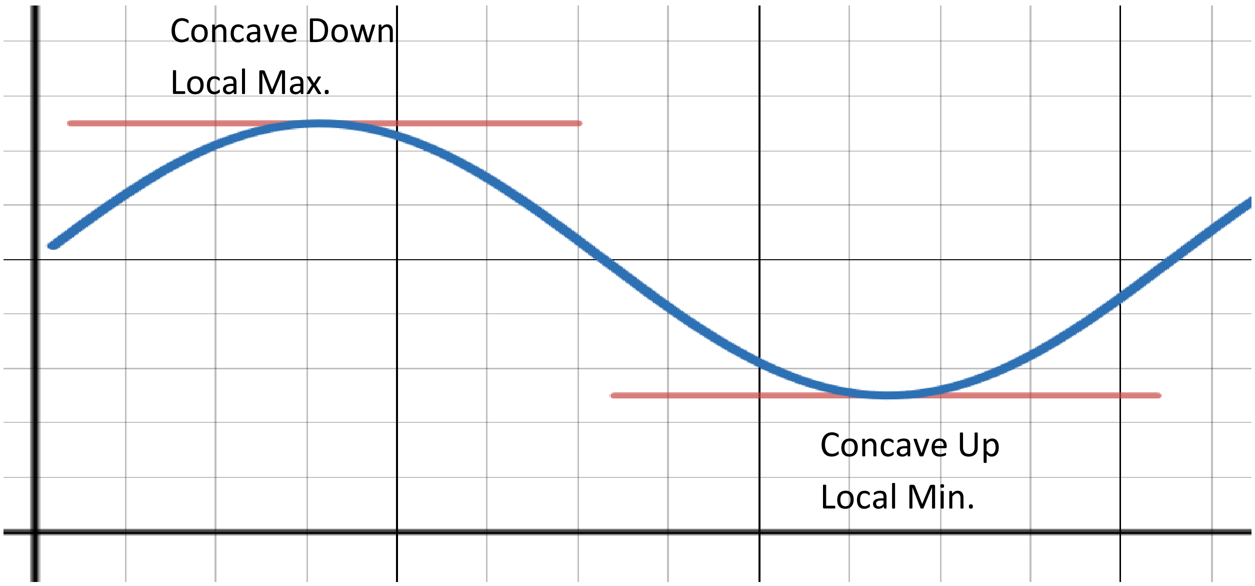



**The Second Derivative Test** (for Local Max./Min): Suppose $f''(x)$ is continuous near $x = c$.

a. If $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has a local Minimum at $x = c$.

b. If $f'(c) = 0$ and $f''(c) < 0$, then $f(x)$ has a local Maximum at $x = c$.

Notice if $f'(c) = 0$ and $f''(c) = 0$, the Second Derivative Test doesn't tell us anything about whether $x = c$ is a relative max or relative min, or neither. In that case we would need to use the First Derivative Test for Max./Min. The reason the 2nd Derivative test is useful is that it is sometimes easier to use than the first derivative test.




Ex. What does the Second Derivative Test tell us about the local max/min of $f(x) = x^4 - 4x^3$?

$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) = 0 \quad \Longrightarrow \quad x = 0, 3.$

$f''(x) = 12x^2 - 24x$

$f''(0) = 12(0)^2 - 24(0) = 0;$ $2^{nd}$ Derivative test fails since $f''(0) = 0.$

$f''(3) = 12(3)^2 - 24(3) = 108 - 72 = 36 > 0$

So by the $2^{nd}$ Derivative test $x = 3,\ y = (3)^4 - 4(3)^3 = -27,$ is a relative minimum.

We saw earlier that:

sign of $f'(x)$ ____ − ____|____ − ____|____ + ____ .

0 3

So by the first derivative test $x = 0$ is neither a local maximum or minimum.

Ex. Use the 2nd Derivative Test to find all relative max/min of $f(x) = x^4 - 2x^2$.

$$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1) = 0$$

$$\Rightarrow \quad x = 0,\ 1, -1.$$

$$f''(x) = 12x^2 - 4$$

$f''(0) = 12(0)^2 - 4 = -4 < 0 \ \Rightarrow x = 0,\ y = 0$ is a relative maximum.

$f''(1) = 12(1)^2 - 4 = 8 > 0 \ \Rightarrow x = 1,\ y = -1$ is a relative minimum.

$f''(-1) = 12(-1)^2 - 4 = 8 > 0 \ \Rightarrow x = -1,\ y = -1$ is a relative minimum.