Related Rate Problems

Related rate problems are problems where you are told about the rate at which some quantity (or quantities) is (are) changing (i.e., you are given the value of a derivative of a quantity with respect to time) and are asked how the rate (i.e., a different derivative with respect to time) of some other quantity is changing.

Steps to Solve a Related Rate Problem:

1. Draw a picture if possible

2. Assign letters to quantities that are changing (these are functions of time)

3. Write down the rates that are given in the problem (these are derivatives with respect to time) and the rate you want to find.

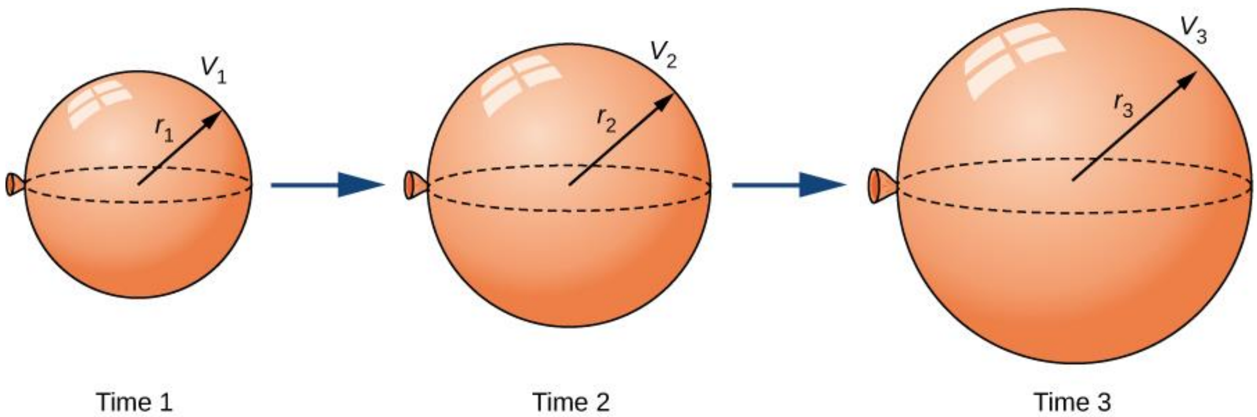
4. Find an equation relating the quantities whose rates you know with the quantities whose rate you want to know (often this equation comes from the pythagorean theorem, similar triangles, or volume/area formulas).

5. Differentiate the equation you just wrote with respect to time (t).

6. Substitute values for the known rates or values of the quantities themselves into the differentiated equation.

7. Solve for the rate (a derivative with respect to time) you want to know.

Ex. Air is being pumped into a spherical balloon at a rate of 20 cu.in./min. What is the rate of change of the radius at the moment the diameter is 10 in.?



$V(t) = Volume \qquad r(t) = radius$

$$\frac{dV}{dt} = 20\ cu.in./min.$$

$\frac{dr}{dt} = ?$ at diameter= $10\ in. \implies r = 5in.$

$V(t) = \frac{4}{3}\pi\big(r(t)\big)^3;$ Equation relates $V(t)$ to $r(t)$. Differentiate:

$\frac{dV}{dt} = 4\pi\big(r(t)\big)^2 \frac{dr}{dt};$ Now plug in known values: $\frac{dV}{dt} = 20; \ r = 5.$

$20 = 4\pi(5)^2(\frac{dr}{dt});$ Now solve for $\frac{dr}{dt}$.

$$20 = 100\pi(\frac{dr}{dt})$$

$$\frac{1}{5\pi}\ in./min = \frac{dr}{dt}.$$

Note: A common mistake is plugging in given values in the problem before differentiating the equation. In related rate problems always differentiate the equation before plugging in given values of the variables and derivatives.

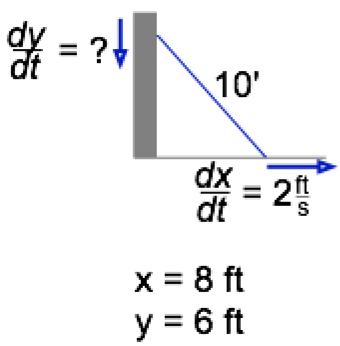
Ex. A ladder 10ft. long rests against a vertical wall. Suppose the bottom of the ladder slides away from the wall at 2 ft/sec.

a. How fast is the top of the ladder sliding down the wall when the bottom of the ladder is 8ft from the wall?

b. At what rate is the area of the triangle created by the ladder, wall, and ground changing when the foot of the ladder is 8 ft. from the wall?

c. At what rate is the angle the ladder makes with the ground changing when the ladder is 8 ft from the wall?

a.



As the ladder slides down the wall

x and y are changing with time.

$$\frac{dx}{dt} = 2ft/sec$$

$\frac{dy}{dt} = ?$ when $x(t) = 8ft.$

$\left(x(t)\right)^2 + \left(y(t)\right)^2 = 10^2;$ Pythagorean identity gives this equation.

Now differentiate the equation with respect to time:

$$2\big(x(t)\big)\frac{dx}{dt} + 2\big(y(t)\big)\frac{dy}{dt} = 0.$$

Now plug in known values: $x = 8,\ y = 6,\ \frac{dx}{dt} = 2.$

$$2(8)(2) + 2(6)\frac{dy}{dt} = 0 \quad \Longrightarrow \quad \frac{dy}{dt} = -\tfrac{8}{3}\,ft/sec.$$

b. Given that $\frac{dx}{dt} = 2ft/sec,$ and $\frac{dy}{dt} = -\frac{8}{3}ft/sec$ when $x = 8ft,$

which we know from part a, find $\frac{dA}{dt}$ when $x = 8.$

So we need an equation that relates the quantities whose derivatives we know (i.e., x, y), with the quantity whose derivative we want to know (i.e., A=area of triangle).

We get this equation from the formula for the area of a triangle:

$$A(t) = \tfrac{1}{2}\big(x(t)\big)(y(t)).$$

Now differentiate with respect to t remembering to use the product rule.

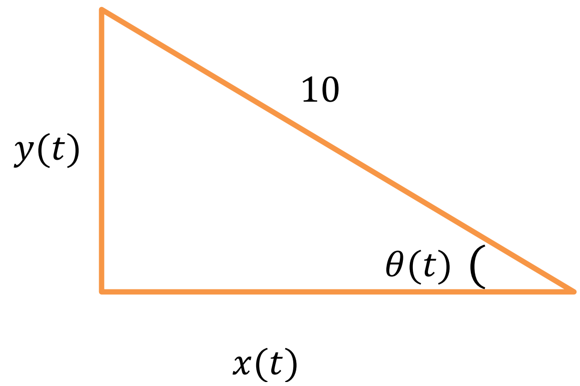
$$\frac{dA(t)}{dt} = \frac{1}{2}\left[\big(x(t)\big)\frac{dy}{dt} + (y(t))\frac{dx}{dt}\right];$$

Now plug in known values: $x = 8,\ \ y = 6,\ \ \frac{dx}{dt} = 2,\ \frac{dy}{dt} = -\frac{3}{8}.$

$$\frac{dA(t)}{dt} = \tfrac{1}{2}[(8)(-\tfrac{8}{3}) + (6)(2)]$$

$$= \tfrac{1}{2}\left(-\tfrac{64}{3} + 12\right) = -\tfrac{14}{3}ft^2/sec.$$

c.



$$\frac{dx}{dt} = 2ft/sec$$

$$\frac{d\theta}{dt} = ?$$ when $x = 8$.

We need to find a relationship between x, whose derivative we know, and θ, whose derivative we want to know. We can get this using trigonometry:

$$\frac{x(t)}{10} = \cos(\theta(t));$$

Now differentiate with respect to t.

$$\frac{1}{10}\frac{dx}{dt} = -(sin(\theta(t)))\frac{d\theta}{dt};$$

Now plug in known values: $x = 8,\ y = 6 \Longrightarrow \sin(\theta(t)) = \frac{6}{10}$.

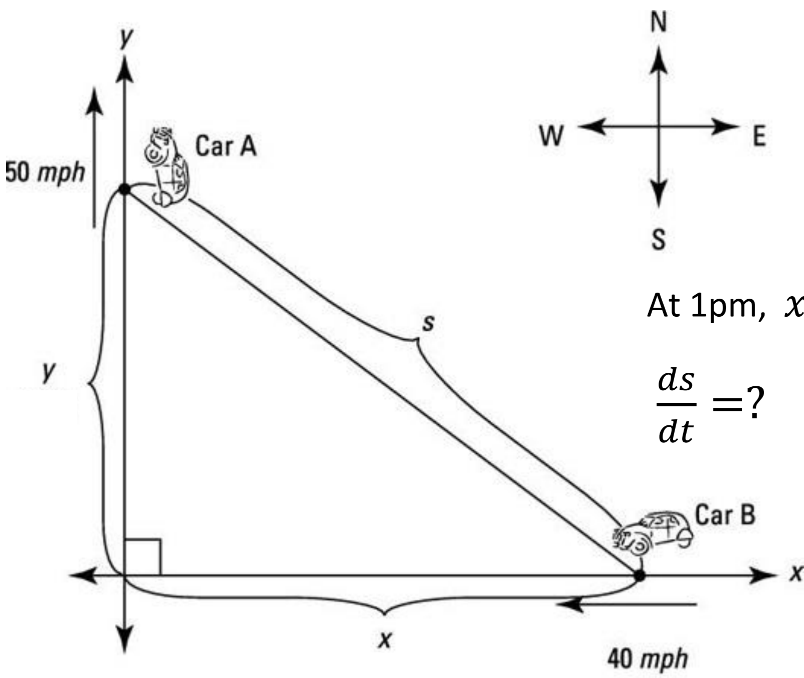
$$\frac{1}{10}(2) = -(\frac{6}{10}))\frac{d\theta}{dt}$$

$$-\frac{1}{3}radians/sec = \frac{d\theta}{dt}.$$

Ex. a. At noon, car A is 10 miles directly north of an intersection travelling north at 50 mph. At same time car B is 120 miles directly east of the intersection travelling west at 40 mph. How fast is the distance between the two cars changing at 1pm?

b. Suppose that car B stops at 1pm, but car A continues north at 50mph. How fast is the distance between the two cars changing at 2pm?

a.



$$\frac{dx}{dt} = -40mph$$

$$\frac{dy}{dt} = 50mph$$

At 1pm, $x = 80,\ y = 60.$

$\frac{ds}{dt} = ?$ when $x = 80,\ \ y = 60.$

So we need to find a relationship between the quantities whose derivatives we know (x, y) and the quantity whose derivative we want to know (s).

$$\big(x(t)\big)^2 + \big(y(t)\big)^2 = \big(s(t)\big)^2;$$ Now differentiate with respect to t.

$$2\big(x(t)\big)\frac{dx}{dt} + 2\big(y(t)\big)\frac{dy}{dt} = 2\big(s(t)\big)\frac{ds}{dt};$$ Plug in known values:

$$x = 80,\ \ y = 60\ \Longrightarrow s = 100\,,\ \ \frac{dx}{dt} = -40,\ \ \frac{dy}{dt} = 50.$$

$$2(80)(-40) + 2(60)50 = 2(100)\frac{ds}{dt}$$

$$-6400 + 6000 = 200\frac{ds}{dt} \qquad \Rightarrow \; -2mph = \frac{ds}{dt}.$$

b. At 2pm car B has stopped and thus $\frac{dx}{dt} = 0mph$ while $x = 80$.

At 2pm car A still has $\frac{dy}{dt} = 50mph$, but is now 110 miles north of the intersection, thus $y = 110$.

We need to find $\frac{ds}{dt}$ when $x = 80, \; y = 110$.

$$x = 80, \; y = 110 \implies s = \sqrt{80^2 + 110^2} = 10\sqrt{185}$$

The relationship is the same as in part a.

$(x(t))^2 + (y(t))^2 = (s(t))^2$; Now differentiate with respect to t.

$$2(x(t))\frac{dx}{dt} + 2(y(t))\frac{dy}{dt} = 2(s(t))\frac{ds}{dt};$$

Plug in known values:

$$x = 80, \; y = 110 \implies s = 10\sqrt{185}, \; \frac{dx}{dt} = 0, \; \frac{dy}{dt} = 50.$$

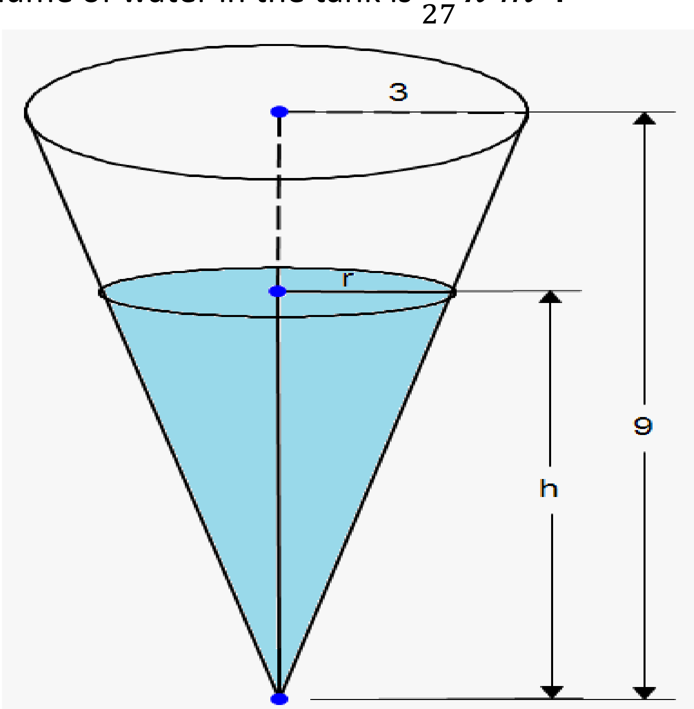
$$2(80)(0) + 2(110)50 = 2\left(10\sqrt{185}\right)\frac{ds}{dt} \Rightarrow$$

$$\frac{ds}{dt} \approx 40.44 \; mph.$$

Ex. a. A water tank has the shape of an inverted circular cone with the radius of the base 3m, and the height 9m. If water is being pumped into the tank at $2m^3/min$, find the rate at which the water level is rising when the water is 6m deep.

b. Assume the same dimensions of the tank. If water is being pumped into the tank at $2m^3/min$, find the rate at which the water level is rising when the volume of water in the tank is $\frac{8}{27}\pi\, m^3$.

a.



$$\frac{dV(t)}{dt} = 2m^3/min$$

$\frac{dh(t)}{dt} =?$ when $h = 6m$.

The volume formula for a circular cone gives us a relationship between the volume, the height and the radius of the cone, $V(t) = \frac{1}{3}\pi\big(r(t)\big)^2(h(t))$.

However, we don't know $\frac{dr(t)}{dt}$. We can either find a relationship between $r(t)$ and $h(t)$ and use it to get a relationship between just the volume and the height, or we can an expression for $\frac{dr(t)}{dt}$ in terms of $\frac{dh(t)}{dt}$. In either case we get this from similar triangles.

From the picture above we get:

$$\frac{r}{h} = \frac{3}{9} \implies r = \frac{1}{3}h\,.$$

Substituting $r = \frac{1}{3}h$ into the volume formula we get:

$$V(t) = \frac{1}{3}\pi\big(r(t)\big)^2(h(t))$$

$$V(t) = \frac{1}{3}\pi\left(\frac{1}{3}h(t)\right)^2\big(h(t)\big) = \frac{\pi}{27}\big(h(t)\big)^3;$$ Now differentiate:

$$\frac{dV(t)}{dt} = \frac{1}{9}\pi\big(h(t)\big)^2\frac{dh(t)}{dt};$$ Now plug in $\frac{dV(t)}{dt} = 2, \;\; h = 6.$

$$2 = \frac{1}{9}\pi(6)^2\frac{dh(t)}{dt}$$

$$2 = 4\pi\frac{dh(t)}{dt} \qquad \implies \qquad \frac{1}{2\pi}m/min. = \frac{dh(t)}{dt}\,.$$

b. $\frac{dV(t)}{dt} = 2m^3/min$

$\frac{dh(t)}{dt} = ?$ when $V = \frac{8\pi}{27}m.$

The initial steps to solve this problem are the same as part a until we get to:

$$\frac{dV(t)}{dt} = \frac{1}{9}\pi\big(h(t)\big)^2\frac{dh(t)}{dt} \qquad (*)$$

Now we have to find h when $V = \frac{8\pi}{27}$.

Since we found that $V(t) = \frac{\pi}{27}\big(h(t)\big)^3$ we have:

$$\frac{8\pi}{27} = \frac{\pi}{27}(h)^3 \implies 8 = h^3 \implies h = 2\,.$$

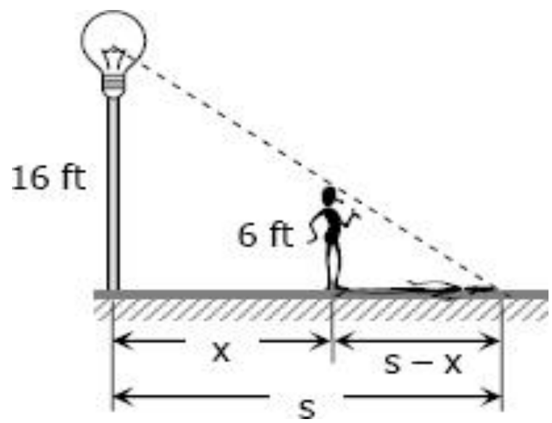
Plugging into $(*)$ we get:

$$2 = \frac{1}{9}\pi(2)^2\frac{dh(t)}{dt} \quad \Longrightarrow \quad \frac{9}{2\pi} m/min. = \frac{dh(t)}{dt}.$$

Ex. A man 6 feet tall walks at a rate of 5 feet per second away from a light that is 16 feet above the ground. When he is 10 feet away from the base of the light,

a. At what rate is the length of his shadow moving?
b. At what rate is the tip of his shadow moving?

a.



$$\frac{dx}{dt} = 5ft/sec$$

$$\frac{d(s-x)}{dt} = ? \quad \text{when } x = 10.$$

To get a relationship between x, whose derivative we know, and $s - x$,

whose derivative we want to know, we use similar triangles:

$$\frac{s-x}{s} = \frac{6}{16}.$$

Now let's solve this equation for s in terms of x:

$$16(s-x) = 6s$$

$$10s = 16x$$

$$s = \frac{8}{5}x.$$

So we have: $s(t) = \frac{8}{5}x(t);$ Now differentiate:

$$\frac{ds}{dt} = \frac{8}{5}\frac{dx}{dt};$$ Now plug in $\frac{dx}{dt} = 5$:

$$\frac{ds}{dt} = \frac{8}{5}(5) = 8ft/sec.$$

But we are looking for:

$$\frac{d(s-x)}{dt} = \frac{ds}{dt} - \frac{dx}{dt} = 8 - 5 = 3ft/sec.$$

b. Rate of tip of shadow $= \frac{ds}{dt} = 8ft/sec$ from part a.