Implicit Differentiation

Often we have functions that are explicitly given as y in terms of x like $y = \sqrt{x^2+1}$ or $y = x(sinx)$. But sometimes the relation between y and x is given implicitly by an equation like $x^2 + y^2 = 1$ or $y^2 = x$. **Sometimes it's not even possible to solve for y in terms of x.** The points that satisfy these equations form a curve in the plane. The question is, how do we find the slope of the tangent line (if it exists) to a point on one of these curves? To find $\frac{dy}{dx}$ in this case (which is the slope of the tangent line at each point (x, y) on the curve) we need to do it through **implicit differentiation**.

Ex. Find $\frac{dy}{dx}$ for the circle given by $x^2 + y^2 = 25$. Find an equation for the tangent line at $(-3,4)$. Where is the tangent line horizontal?

To do this we treat y as a function of x, ie $y(x)$, and differentiate the equation using the chain rule. We differentiate any pure expressions in x using our differentiation rules.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + \frac{d}{dx}(y^2) = 0.$$

To calculate $\frac{d}{dx}(y^2)$ treat y as $y(x)$ and use the chain rule.

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(y(x)^2) = 2\big(y(x)\big)y'(x) = 2y(x)\frac{dy}{dx}.$$

So we have: $2x + 2y\frac{dy}{dx} = 0.$

Now solve this equation for $\frac{dy}{dx}$: $\quad 2y\frac{dy}{dx} = -2x$

$$\frac{dy}{dx} = -\frac{x}{y}\,.$$

So the slope of the tangent line at $(-3,4)$ is $\frac{3}{4}$,

An equation of the tangent line at $(-3,4)$ is: $\quad y - 4 = \frac{3}{4}(x + 3).$

The tangent line is horizontal when $\frac{dy}{dx} = 0.$

$\frac{dy}{dx} = \frac{-x}{y} = 0$ when $x = 0.$

Solving for y, we get: $0^2 + y^2 = 25$ means $y = -5, 5.$

So horizontal tangent lines at $(0,-5), (0,5).$

So when differentiating an equation with x's and y's in it, differentiate the terms with just x in them as we have before. Every time you differentiate a term with a y in it, you will get a $\frac{dy}{dx}$ coming from the chain rule.

Ex. Find $\frac{dy}{dx}$ for the equation $x^3 + y^3 = 9$. Find an equation of the tangent line at $(1,2)$. How do we know that $(1,2)$ is on this curve?

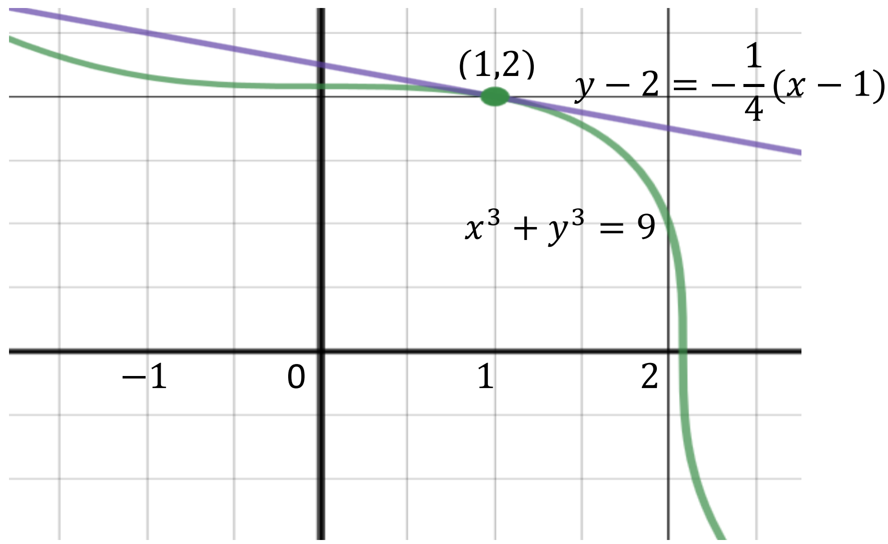
$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(9)$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(9)$$

$$3x^2 + (3y^2)\frac{dy}{dx} = 0$$

$$(3y^2)\frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = -\frac{x^2}{y^2}.$$



At $(1,2)$; $\quad \frac{dy}{dx} = -\frac{(1)^2}{2^2} = -\frac{1}{4}.$

Equation of tangent line at $(1,2)$: $\quad y - 2 = -\frac{1}{4}(x - 1).$

$(1,2)$ is on this curve if $x = 1,\ y = 2$ satisfies the equation $x^3 + y^3 = 9$.

$$(1)^3 + (2)^3 = 1 + 8 = 9,$$

so $(1,2)$ is on the curve with the equation $x^3 + y^3 = 9$.

Ex. Find all points on the curve $x + y^3 - 3y = 1$ where the tangent line is vertical (ie $\frac{dy}{dx}$ is infinite).

$$\frac{d}{dx}(x + y^3 - 3y) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}(x) + \frac{d}{dx}(y^3) - \frac{d}{dx}(3y) = \frac{d}{dx}(1)$$

$$1 + 3y^2\frac{dy}{dx} - 3\frac{dy}{dx} = 0$$

$$1 + (3y^2 - 3)\frac{dy}{dx} = 0$$

$$(3y^2 - 3)\frac{dy}{dx} = -1 \quad \text{or} \quad \frac{dy}{dx} = \frac{-1}{3y^2-3}.$$

$\frac{dy}{dx}$ will be infinite when $3y^2 - 3 = 0$.

$3y^2 - 3 = 3(y^2 - 1) = 3(y - 1)(y + 1) = 0$

So $y = -1, 1$.

Find the x coordinate by plugging back into the original equation:

When $y = -1$:

$x + y^3 - 3y = 1$

$x + (-1)^3 - 3(-1) = 1$

$x - 1 + 3 = 1$

$x = -1.$

When $y = 1$:

$x + y^3 - 3y = 1$

$x + 1^3 - 3(1) = 1$

$x + 1 - 3 = 1$

$x = 3.$

So the points with vertical tangents are: $(-1, -1)$ and $(3,1)$.

Ex. Find an equation of the tangent line to $x^3 + y^3 = 6xy - 1$ at $(3,2)$.

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(6xy - 1)$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(6xy) - \frac{d}{dx}(1)$$

$$3x^2 + 3y^2\frac{dy}{dx} = \frac{d}{dx}(6xy) - 0. \qquad (*)$$

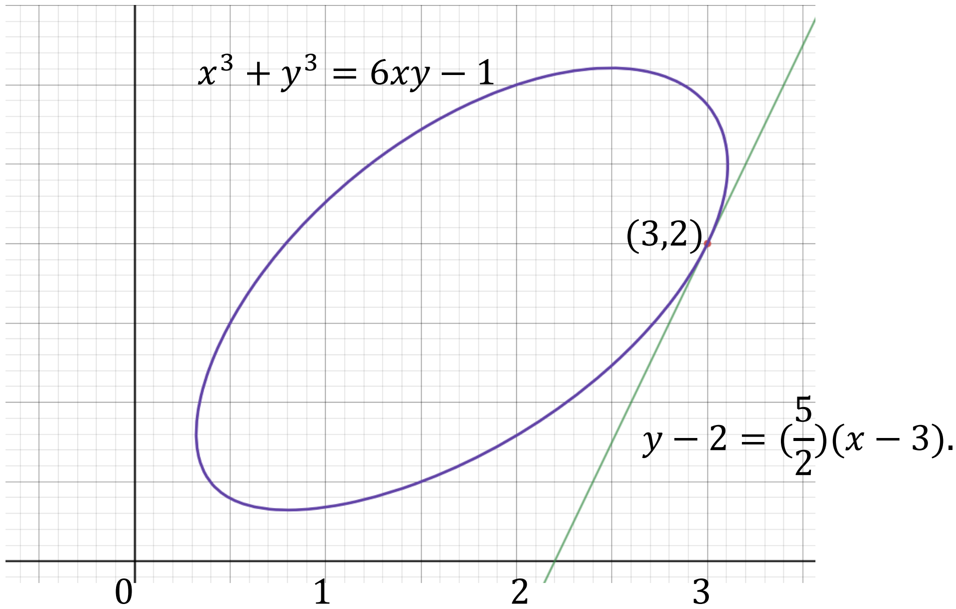
We need to be careful calculating $\frac{d}{dx}(6xy)$, because xy is a product of 2 functions of x so we need to use the product rule.

$$\frac{d}{dx}(6xy) = 6\left[(x)\left(\frac{dy}{dx}\right) + y\frac{d}{dx}(x)\right] = 6x\frac{dy}{dx} + 6y,$$ now plug this into $(*)$

$$3x^2 + 3y^2\frac{dy}{dx} = 6x\frac{dy}{dx} + 6y.$$ Now:

solve for $\frac{dy}{dx}$	**OR**
$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$	just plug in $x = 3$ and $y = 2$ and solve for $\frac{dy}{dx}$
$(3y^2 - 6x)\frac{dy}{dx} = 6y - 3x^2$	$3x^2 + 3y^2\frac{dy}{dx} = 6x\frac{dy}{dx} + 6y$
$\frac{dy}{dx} = \frac{6y-3x^2}{3y^2-6x}$	$3(3)^2 + 3(2)^2\frac{dy}{dx} = 6(3)\frac{dy}{dx} + 6(2)$
At the point $(3,2)$ we get	$27 + 12\frac{dy}{dx} = 18\frac{dy}{dx} + 12$
$\frac{dy}{dx} = \frac{6(2)-3(3)^2}{3(2)^2-6(3)} = \frac{12-27}{12-18} = \frac{5}{2}$	$15 = 6\frac{dy}{dx}$, so $\frac{dy}{dx} = \frac{5}{2}$

Equation of tangent line at $(3,2)$: $\quad y - 2 = (\frac{5}{2})(x - 3).$



Ex. Find $\frac{dy}{dx}$ for the equation $y - (x^2y - 1)^4 = 1$ at the point $(1,2)$.

$$\frac{d}{dx}\left(y - \left(x^2y - 1\right)^4\right) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}(y) - \frac{d}{dx}(x^2y - 1)^4 = 0$$

$$\frac{dy}{dx} - 4(x^2y - 1)^3 \frac{d}{dx}(x^2y - 1) = 0$$

Now use the product rule.

$$\frac{dy}{dx} - 4(x^2y - 1)^3 (x^2 \frac{dy}{dx} + y\frac{d}{dx}(x^2) - \frac{d}{dx}(1)) = 0$$

$$\frac{dy}{dx} - 4(x^2y - 1)^3 (x^2 \frac{dy}{dx} + y(2x) - 0) = 0$$

$\frac{dy}{dx} - 4(x^2y-1)^3(x^2\frac{dy}{dx} + 2xy) = 0.$ Now plug in $(1{,}2)$ and solve for $\frac{dy}{dx}$.

$$\frac{dy}{dx} - 4((1)^2(2)-1)^3\left((1)^1\frac{dy}{dx} + 2(1)(2)\right) = 0$$

$$\frac{dy}{dx} - 4(1)^3\left(\frac{dy}{dx} + 4\right) = 0$$

$$\frac{dy}{dx} - 4\frac{dy}{dx} - 16 = 0$$

$$-3\frac{dy}{dx} = 16 \Longrightarrow \quad \frac{dy}{dx} = -\frac{16}{3}.$$

Ex. Find an equation of the tangent line to $y = \sin(xy^2)$ at $(\frac{\pi}{2}, 1)$.

$\frac{d}{dx}(y) = \frac{d}{dx}(\sin(xy^2))$; By the chain rule $\frac{d}{dx}\left(sinu(x)\right) = (cosu(x))\frac{du}{dx}$.

$\frac{dy}{dx} = (\cos(xy^2))\frac{d}{dx}(xy^2)$ Now use the product rule.

$$\frac{dy}{dx} = (\cos(xy^2))(x\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x))$$

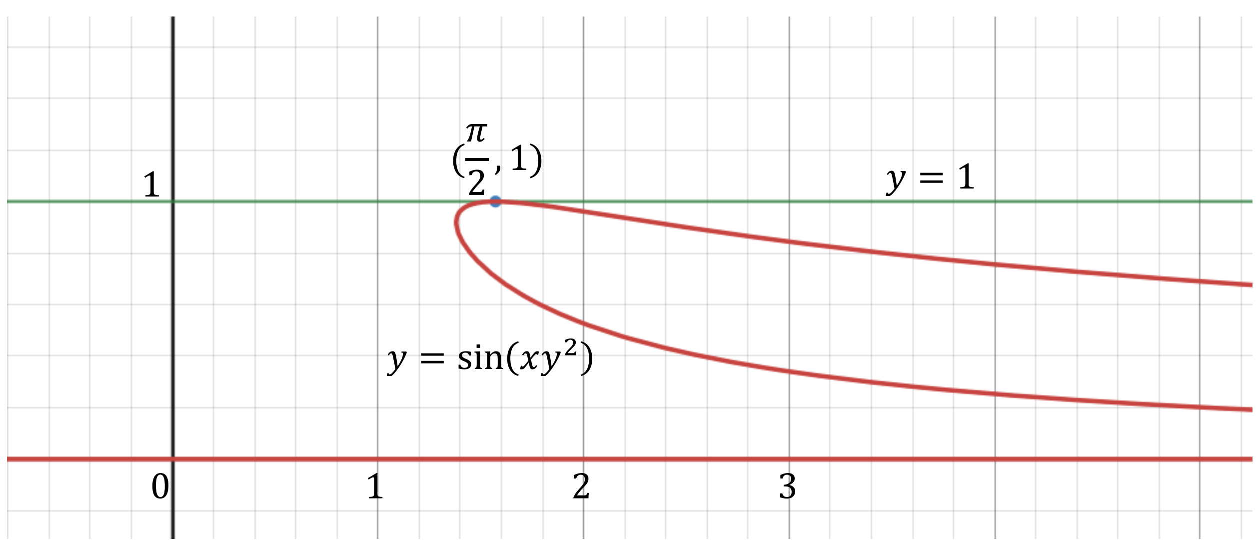
$$\frac{dy}{dx} = (\cos(xy^2))(x\,(2y\frac{dy}{dx}) + y^2(1))$$

$$\frac{dy}{dx} = (\cos(xy^2))(2xy\frac{dy}{dx} + y^2)$$ Now plug in $(\frac{\pi}{2}, 1)$.

$$\frac{dy}{dx} = \left(\cos\left(\left(\frac{\pi}{2}\right)(1^2)\right)\right)\left(2\left(\frac{\pi}{2}\right)(1)\frac{dy}{dx} + 1^2\right);$$ $\cos\left(\frac{\pi}{2}\right) = 0$ so

$$\frac{dy}{dx} = 0\,.$$

Equation of tangent line at $(\frac{\pi}{2}, 1)$: $y = 1.$



The power rule for rational exponents:

Suppose that m and n are integers with $n \neq 0$. Then

$$\frac{d}{dx}\left(x^{\frac{m}{n}}\right) = \frac{m}{n} x^{\left(\frac{m}{n}-1\right)}$$

as long as $x > 0$ when n is even.

Ex. Calculate $\frac{dy}{dx}$ for

a. $y = \frac{1}{\sqrt[4]{x}}$

b. $y = (x + sinx)^{\frac{2}{3}}$

$a.$ $y = \frac{1}{\sqrt[4]{x}} = x^{-\frac{1}{4}}$ (easier to do this than using the quotient rule)

$$\frac{dy}{dx} = -\frac{1}{4} x^{-\frac{5}{4}}$$

b. $\frac{dy}{dx} = \frac{2}{3}(x + sinx)^{-\frac{1}{3}} \frac{d}{dx}(x + sinx)$

$$= \frac{2}{3}(x + sinx)^{-\frac{1}{3}}(1 + cosx).$$

Ex. Find the slope of the curve $2(x+y)^{\frac{1}{3}} = y$ at the point $(4,4)$.

$$\frac{d}{dx}\left(2\left(x+y\right)^{\frac{1}{3}}\right) = \frac{d}{dx}(y)$$

$$\frac{2}{3}(x+y)^{-\frac{2}{3}}\left(\frac{d}{dx}(x+y)\right) = \frac{dy}{dx}$$

$$\frac{2}{3}(x+y)^{-\frac{2}{3}}\left(1+\frac{dy}{dx}\right) = \frac{dy}{dx}$$

$$\frac{2}{3}(x+y)^{-\frac{2}{3}} + \frac{2}{3}(x+y)^{-\frac{2}{3}}\left(\frac{dy}{dx}\right) = \frac{dy}{dx}$$

Now plug in $x = 4,\; y = 4$ and solve for $\dfrac{dy}{dx}$:

$$\frac{2}{3}(4+4)^{-\frac{2}{3}} + \frac{2}{3}(4+4)^{-\frac{2}{3}}\left(\frac{dy}{dx}\right) = \frac{dy}{dx}$$

$$\frac{2}{3}(8)^{-\frac{2}{3}} + \frac{2}{3}(8)^{-\frac{2}{3}}\left(\frac{dy}{dx}\right) = \frac{dy}{dx}$$

$$\frac{2}{3}\left(\frac{1}{(\sqrt[3]{8})^2}\right) + \frac{2}{3}\left(\frac{1}{(\sqrt[3]{8})^2}\right)\left(\frac{dy}{dx}\right) = \frac{dy}{dx}$$

$$\frac{2}{3}\left(\frac{1}{2^2}\right) + \frac{2}{3}\left(\frac{1}{2^2}\right)\left(\frac{dy}{dx}\right) = \frac{dy}{dx}$$

$$\frac{1}{6} + \frac{1}{6}\left(\frac{dy}{dx}\right) = \frac{dy}{dx}$$

$$\frac{1}{6} = \frac{5}{6}\left(\frac{dy}{dx}\right)$$

$$\frac{1}{6}\left(\frac{6}{5}\right) = \frac{dy}{dx}$$

$$\frac{1}{5} = \frac{dy}{dx}.$$

Ex. Find an equation of the tangent line to $x(\sqrt[3]{y}) + y = 10$ at $(1,8)$.

$$\frac{d}{dx}\left(x(\sqrt[3]{y}) + y\right) = \frac{d}{dx}(10);$$ Now use the product rule.

$$x\frac{d}{dx}\left(y^{\frac{1}{3}}\right) + y^{\frac{1}{3}}\frac{d}{dx}(x) + \frac{d}{dx}(y) = 0$$

$$x\left(\frac{1}{3}y^{-\frac{2}{3}}\frac{dy}{dx}\right) + y^{\frac{1}{3}}(1) + \frac{dy}{dx} = 0$$

$$\frac{1}{3}x\left(\frac{1}{\sqrt[3]{y^2}}\right)\frac{dy}{dx} + \sqrt[3]{y} + \frac{dy}{dx} = 0;$$ Now plug in $(1,8)$ and solve for $\frac{dy}{dx}$.

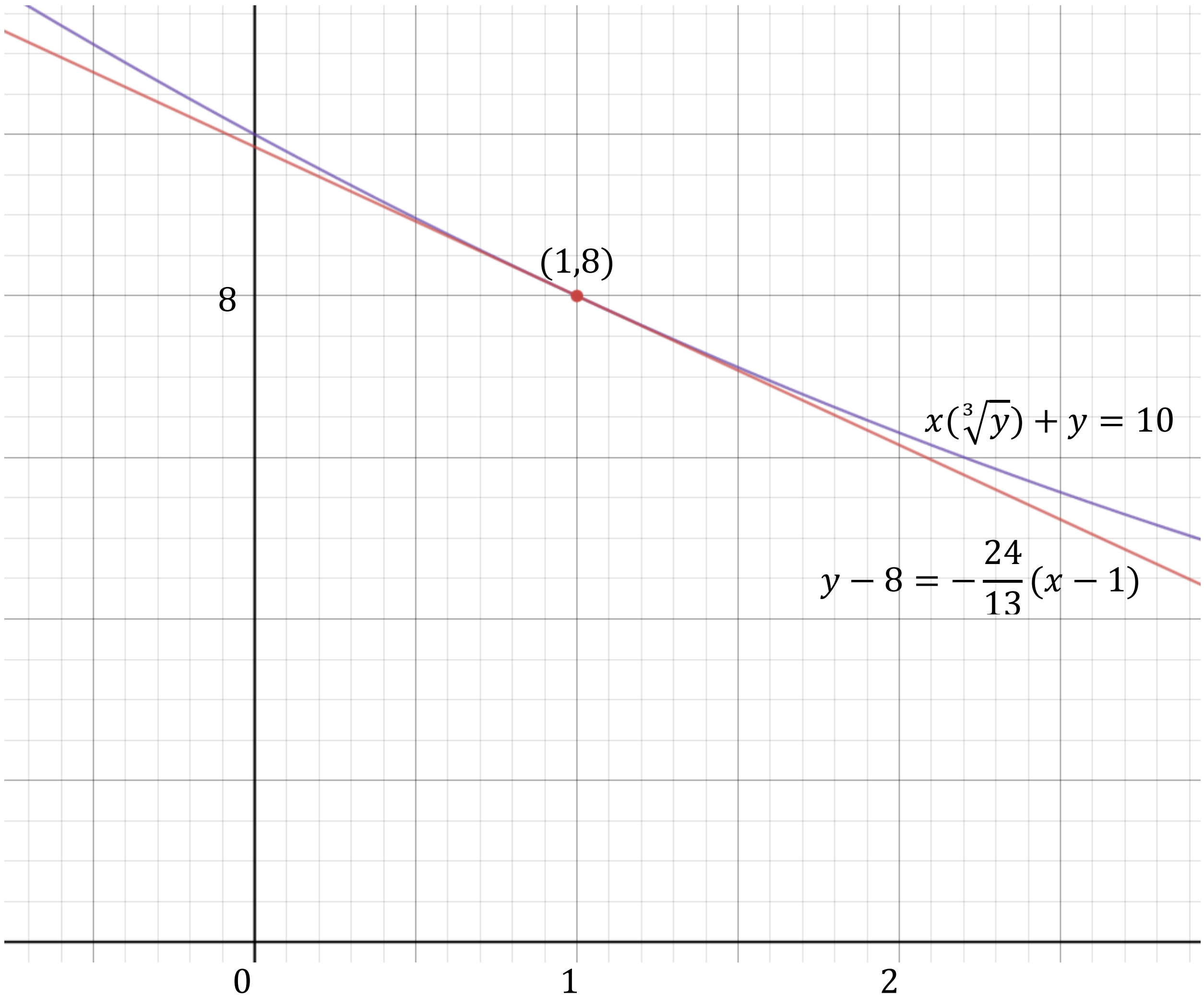
$$\frac{1}{3}(1)\left(\frac{1}{\sqrt[3]{64}}\right)\frac{dy}{dx} + \sqrt[3]{8} + \frac{dy}{dx} = 0$$

$$\frac{1}{3}\left(\frac{1}{4}\right)\frac{dy}{dx} + 2 + \frac{dy}{dx} = 0$$

$$\frac{1}{12}\frac{dy}{dx} + 2 + \frac{dy}{dx} = 0$$

$$\frac{13}{12}\frac{dy}{dx} = -2 \implies \frac{dy}{dx} = -\frac{24}{13}.$$

Equation of tangent line at $(1,8)$: $y - 8 = -\frac{24}{13}(x - 1)$.


(1,8)
8
$x(\sqrt[3]{y}) + y = 10$
$y - 8 = -\frac{24}{13}(x - 1)$
0
1
2