The Chain Rule

The chain rule is a formula for differentiating the composition of 2 functions.

Ex. Let $f(x) = x^{100}$ and let $g(x) = 5x + 2$ then

$$f\big(g(x)\big) = f(5x+2) = (5x+2)^{100}.$$

At the moment we have no easy way to differentiate this function.

Chain Rule: Suppose $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at x. The composition function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed in two equivalent ways:

1. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

2. $\frac{d}{dx}(f\big(g(x)\big) = f'\big(g(x)\big)g'(x).$

When using the chain rule it's important to be able to identify the "inside" function $g(x)$. In the case of $h(x) = (5x+2)^{100}$, the "inside" function is $5x + 2$. One often thinks of the chain rule as saying when we differentiate a composite function, we differentiate the "function" (in this example x^{100}) and multiply that derivative by the derivative of the "inside" function.

Ex. Differentiate $h(x) = (5x+2)^{100}$.

We can solve this a couple of ways:

1. $y = f(u) = u^{100}$, and $u = g(x) = 5x + 2$

$$\frac{dy}{du} = 100u^{99}, \qquad \frac{du}{dx} = 5.$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{dy}}{\mathrm{du}}\frac{\mathrm{du}}{\mathrm{dx}} = (100u^{99})(5) = 100(5x+2)^{99}(5) = 500(5x+2)^{99}$$

2. $h'(x) = 100(5x+2)^{99}\frac{d}{dx}(5x+2)$

$$= 100(5x+2)^{99}(5)$$

$$= 500(5x+2)^{99}.$$

The Chain Rule for Powers of a Function.

If f is differentiable for all x in its domain and n is an integer, then

$$\frac{d}{dx}((f(x))^n) = n(f(x))^{n-1}f'(x).$$

Ex. Write $h(x) = (x^2-4)^{20}$ as a composite function in 2 ways: $y = f(u)$, $u = g(x)$; and $h(x) = f(g(x))$.

1. $u = x^2 - 4$, $y = f(u) = u^{20}$; thus $f(u) = u^{20} = (x^2-4)^{20}$.

2. $g(x) = x^2 - 4$, $f(x) = x^{20}$; thus $f\big(g(x)\big) = (x^2-4)^{20}$.

Ex. Use the 2 forms of the Chain Rule to find the derivative of

$h(x) = (x^2 - 4)^{20}$.

1. $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$y = f(u) = u^{20}$, so $\frac{dy}{du} = 20u^{19}$

$u = x^2 - 4$, so $\frac{du}{dx} = 2x$.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (20u^{19})(2x)$$
$$= 20(x^2 - 4)^{19}(2x) = 40x(x^2 - 4)^{19}.$$

2. $\frac{d}{dx}(f(g(x)) = f'(g(x))g'(x)$

$f(x) = x^{20}$, thus $f'(x) = 20x^{19}$.

$g(x) = x^2 - 4$, thus $f'(g(x)) = 20(x^2 - 4)^{19}$;

and $g'(x) = 2x$.

$$h'(x) = \frac{d}{dx}(f(g(x)) = f'(g(x))g'(x) = (20(x^2 - 4)^{19})(2x)$$
$$= 40x(x^2 - 4)^{19}.$$

Ex. Use the 2 forms of the Chain Rule to find $h'(x)$ if $h(x) = (x^2 + 3x)^{12}$.

1. $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$y = f(u) = u^{12};$ so $\frac{dy}{du} = 12u^{11}$

$u = x^2 + 3x;$ so $\frac{du}{dx} = 2x + 3.$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (12u^{11})(2x + 3)$$

$$= 12(x^2 + 3x)^{11}(2x + 3).$$

2. $\frac{d}{dx}(f(g(x)) = f'(g(x))g'(x)$

$f(x) = x^{12},$ thus $f'(x) = 12x^{11}.$

$g(x) = x^2 + 3x,$ thus $f'(g(x)) = 12(x^2 + 3x)^{11};$

and $g'(x) = 2x + 3.$

$$h'(x) = \frac{d}{dx}(f(g(x)) = f'(g(x))g'(x) = (12(x^2 + 3x)^{11})(2x + 3).$$

Ex. Differentiate

a. $f(x) = \sin(x^2)$

b. $g(x) = sin^2x.$

Note: these are different functions.

a. For $f(x) = \sin(x^2)$ we can think of this as $y = sinu$ and $u = x^2$.

$$f'(x) = \frac{dy}{du}\frac{du}{dx} = (cosu)(2x) = (cos(x^2))(2x).$$

Or we can think of x^2 as the inside function and differentiate the sine function and multiply by the derivative of the inside.

$$f'(x) = (\cos(x^2))\left(\frac{d}{dx}(x^2)\right) = (\cos(x^2))(2x).$$

b. $g(x) = sin^2x = (sinx)^2.$

We can think of this as $y = u^2$ and $u = sinx$.

$$g'(x) = \frac{dy}{du}\frac{du}{dx} = (2u)(cosx) = (2sinx)(cosx).$$

Or we can think of $sinx$ as the “inside” function and differentiate x^2 and multiply by the derivative of $sinx$.

$$g'(x) = 2(sinx)(\frac{d}{dx}(sinx))$$

$$= 2(sinx)(cosx).$$

Ex. Find the derivative of $g(t) = \sec(t^3 + t + 1)$.

$$y = secu, \quad u = t^3 + t + 1.$$

$$g'(t) = \frac{dy}{du}\frac{du}{dt} = (secu)(tanu)(3t^2 + 1)$$
$$= [(sec(t^3 + t + 1))(\tan(t^3 + t + 1))](3t^2 + 1).$$

or

$$g'(t) = (sec(t^3 + t + 1)(\tan(t^3 + t + 1))\frac{d}{dt}(t^3 + t + 1)$$
$$= [(sec(t^3 + t + 1))(\tan(t^3 + t + 1))](3t^2 + 1).$$

Ex. Find the derivative of $y = (cscx + 2\sqrt{x})^5$.

$$y = u^5; \qquad u = cscx + 2\sqrt{x}\,.$$

$$\frac{dy}{du} = 5u^4; \qquad \frac{du}{dx} = -(cscx)(cotx) + \frac{2}{2\sqrt{x}} = -(cscx)(cotx) + \frac{1}{\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 5u^4(-(cscx)(cotx) + \frac{1}{\sqrt{x}})$$
$$= 5\left(cscx + 2\sqrt{x}\right)^4(-(cscx)(cotx) + \frac{1}{\sqrt{x}}).$$

Or

$$\frac{dy}{dx} = 5\left(cscx + 2\sqrt{x}\,\right)^4 \frac{d}{dx}(cscx + 2\sqrt{x}\,)$$

$$= 5\left(cscx + 2\sqrt{x}\right)^4(-(cscx)(cotx) + \frac{1}{\sqrt{x}}).$$

Ex. Find the derivative of $y = x^2\sin(6x)$.

This is a product of functions so we need to start with the product rule.

$$\frac{dy}{dx} = (x^2)\left(\frac{d}{dx}(sin6x)\right) + (sin6x)\frac{d}{dx}(x^2)$$

$$= (x^2)(\cos(6x))\left(\frac{d}{dx}(6x)\right) + (sin6x)(2x)$$

$$= (x^2)(\cos(6x))(6) + 2x(sin6x)$$

$$= 6(x^2)(\cos(6x)) + 2x(sin6x).$$

Ex. Find the derivative of $y = (\frac{cosx}{1-cosx})^4$.

$$y = u^4, \qquad u = \frac{cosx}{1-cosx}$$

$$\frac{dy}{du} = 4u^3; \qquad \frac{du}{dx} = \frac{(1-cosx)\frac{d}{dx}(cosx)-(cosx)\frac{d}{dx}(1-cosx)}{(1-cosx)^2}$$

$$= \frac{(1-cosx)(-sinx)-(cosx)(sinx)}{(1-cosx)^2} = \frac{-sinx}{(1-cosx)^2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 4u^3\left(\frac{-sinx}{(1-cosx)^2}\right) = 4\left(\frac{cosx}{1-cosx}\right)^3\left(\frac{-sinx}{(1-cosx)^2}\right)$$

$$= \frac{-4(\cos^3 x)(sinx)}{(1-cosx)^5}.$$

Ex. Suppose $F(x) = f(g(x))$. Use the table below to find $F'(2)$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	2	3	4	-3
1	3	-1	7	2
2	6	1	5	-2

$F(x) = f(g(x))$,

So by the chain rule: $F'(x) = f'\big(g(x)\big)g'(x)$.

Thus at $x = 2$ we have: $F'(2) = f'\big(g(2)\big)g'(2) \qquad (*)$.

From the table we see that: $g(2) = 1$ and $g'(2) = -2$.

So plugging into $(*)$ we get: $F'(2) = f'(1)g'(2)$

$$= (7)(-2) = -14.$$

Ex. Let $h(x) = \big(3 + 2g(x)\big)^3$. Suppose that $g(4) = -1$ and $g'(4) = 2$.

Find $h'(4)$.

By the chain rule: $h'(x) = 3\big(3 + 2g(x)\big)^2\big(2g'(x)\big)$.

So at $x = 4$ we have: $h'(4) = 3\big(3 + 2g(4)\big)^2(2g'(4))$.

$g(4) = -1$ and $g'(4) = 2 \Rightarrow \quad h'(4) = 3\big(3 + 2(-1)\big)^2(2(2))$

$$= 3(1)^2(4) = 12.$$

Composition of 3 or more functions:

When a function is a composition of 3 or more functions you need to use the chain rule more than once in this problem.

Ex. Write $y = sin^2(x^{10})$ as a composition of 3 functions, $y = f(g(h(x)))$

and then find $\frac{dy}{dx}$.

Let $h(x) = x^{10}$, $g(x) = sinx$, and $f(x) = x^2$,

then $f\left(g(h(x))\right) = sin^2(x^{10})$.

$$y = sin^2(x^{10}) = (\sin(x^{10}))^2$$

$$\frac{dy}{dx} = 2(\sin(x^{10}))\frac{d}{dx}(\sin(x^{10})) = 2(\sin(x^{10}))(\cos(x^{10}))\frac{d}{dx}(x^{10})$$

$$= 2(\sin(x^{10}))(\cos(x^{10}))(10x^9)$$

$$= 20x^9(\sin(x^{10}))(\cos(x^{10})).$$

Ex. write $y = (2 + tan^2x)^5$ as a composition of 3 functions,

$y = f(g(h(x)))$ and then find $\frac{dy}{dx}$.

$$h(x) = tanx\ , \quad g(x) = 2 + x^2, \quad f(x) = x^5$$

$$\frac{dy}{dx} = 5(2 + tan^2x)^4 \frac{d}{dx}(2 + tan^2x)$$

$$= 5(2 + tan^2x)^4(0 + 2tanx \frac{d}{dx}(tanx))$$

$$= 5(2 + tan^2x)^4(2(tanx)(sec^2x))$$

$$= 10(2 + tan^2x)^4(tanx)(sec^2x).$$

Ex. Find $\frac{d^2y}{dx^2}$ if $y = xcos(x^2)$.

We start with the product rule:

$$\frac{dy}{dx} = x\frac{d}{dx}(\cos(x^2)) + (\cos(x^2))\frac{d}{dx}(x)$$

$$= x\frac{d}{dx}(\cos(x^2)) + \cos(x^2)\,;$$ now use the chain rule:

$$= x(-\sin(x^2))\frac{d}{dx}(x^2) + \cos(x^2)$$

$$= x(-\sin(x^2))(2x) + \cos(x^2)$$

$$\frac{dy}{dx} = -2x^2(\sin(x^2)) + \cos(x^2).$$

To find the second derivative we differentiate the first derivative. We need to use the product rule and the chain rule on the first term and the chain rule on the second term:

$$\frac{d^2y}{dx^2} = -2[x^2 \frac{d}{dx}\sin(x^2) + \sin(x^2)\frac{d}{dx}(x^2)] + (-\sin(x^2))(\frac{d}{dx}(x^2))$$

$$= -2[x^2(\cos(x^2))\frac{d}{dx}(x^2) + (\sin(x^2))\,(2x)] - (\sin(x^2))(2x)$$

$$= -2[x^2(\cos(x^2))(2x) + (\sin(x^2))(2x)] - (\sin(x^2))(2x)$$

$$= -4x^3(\cos(x^2)) - 6x(\sin(x^2)).$$