Derivatives and Rates of Change

One Dimensional Motion:

Average Velocity vs Instantaneous Velocity

Let $s(t)$ be the position of an object moving in a line (up/down or right/left).

$s(t+h)-s(t)$ =**displacement** between $[t, t+h]$

$$\frac{s(t+h)-s(t)}{h} = \boldsymbol{average\ velocity}\ between\ [t, t+h].$$

$$\lim_{h\to 0}\frac{s(t+h)-s(t)}{h} = \boldsymbol{instantaneous\ velocity} = v(t) = s'(t).$$

(Note: When $v(t) > 0$ the object is moving to the right or up. When $v(t) < 0$ the object is moving to the left or down. When $v(t) = 0$, it's at rest)

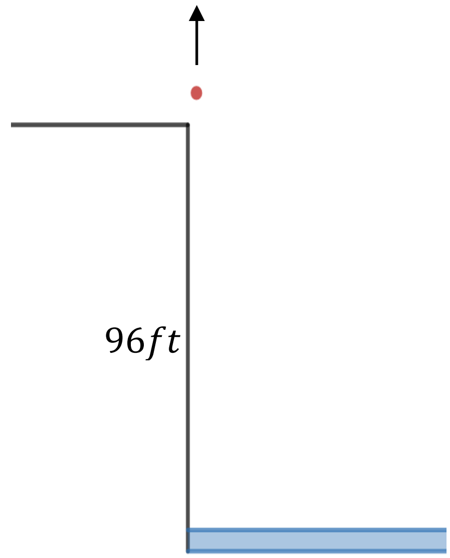
Speed at time $t = |v(t)| = |s'(t)|$

(Note: speed is always non-negative. Unlike velocity, it does not have a direction, only a magnitude)

$$a(t) = \boldsymbol{acceleration}\ at\ time\ t = v'(t) = s''(t).$$

Ex. A ball is thrown vertically into the air at 80 ft/sec from the edge of a cliff 96 ft above the water below. The position of the ball (in feet above the water) at time $t\ sec$ is given by

$$s(t) = -16t^2 + 80t + 96.$$



a. Determine the velocity of the ball at time t.
b. When does the ball reach its highest point?
c. What is the ball's highest point above the water?
d. When does the ball hit the water?
e. With what velocity does the ball hit the water?
f. What is the speed when the ball hits the water?
g. What is the acceleration when the ball hits the water?

a. $v(t) = s'(t) = -32t + 80$ ft/sec .

b. Highest point when $v(t) = s'(t) = 0.$

$$-32t + 80 = 0$$

$$80 = 32t$$

$$t = \frac{80}{32} = \frac{5}{2}\ sec.$$

c. $s\left(\frac{5}{2}\right) = -16(\frac{5}{2})^2 + 80(\frac{5}{2}) + 96$

$$= -16(\frac{25}{4}) + 200 + 96$$

$$= -100 + 200 + 96 = 196ft.$$

d. When is $s(t) = 0$?

$-16t^2 + 80t + 96 = 0$

$-16(t^2 - 5t - 6) = 0$

$-16(t - 6)(t + 1) = 0$

$t = 6, -1.$

Since $t \geq 0$, $t = 6$ sec.

e. $v(t) = -32t + 80$; When the ball hits the water $t = 6$.

$v(6) = -32(6) + 80$

$v(6) = -192 + 80 = -112 ft/sec.$

f. Speed $= |v(t)|$; So the speed when the ball hits the water is

$|v(6)| = |-112| = 112 ft/sec.$

g. $a(t) = v'(t) = s''(t) = -32 ft/sec^2.$

In this example the acceleration is constant at $-32 ft/sec^2$.

Ex. A particle moves according to a law of motion $s = f(t), t \geq 0$, where t is measured in seconds and s in feet. s is given by:

$$s(t) = t^3 - 6t^2 + 9t$$

a. Find the velocity at time t and draw a graph of $v(t)$.

b. What are the velocity and acceleration after 3 secs.?

c. When is the particle at rest?

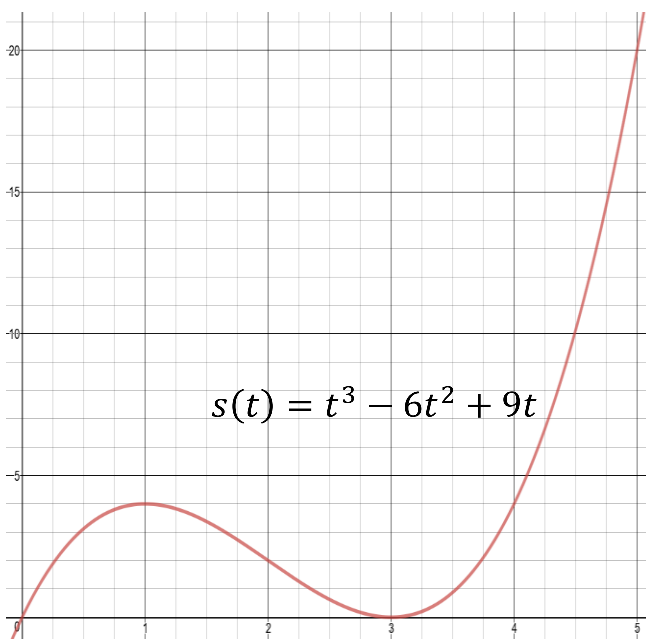
d. When is the particle moving in the positive direction?

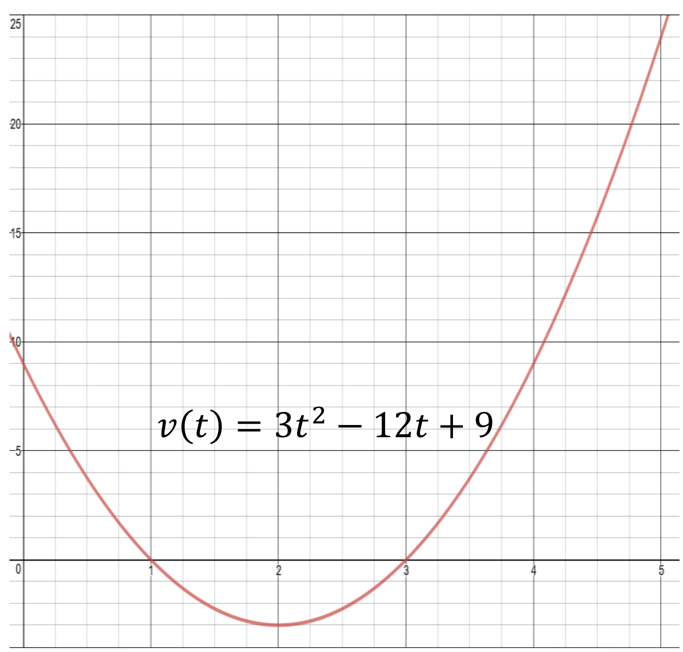
e. Find the total distance traveled during the first 5 secs.

a. $v(t) = s'(t) = 3t^2 - 12t + 9$

$$= 3(t^2 - 4t + 3)$$

$$= 3(t - 3)(t - 1).$$





b. $v(3) = 3(3^2) - 12(3) + 9 = 0\ ft/sec.$

$a(t) = v'(t) = 6t - 12.$

$a(3) = 6(3) - 12 = 6ft/sec^2.$

c. Particle is at rest when $v(t) = 0.$ From part a we know:

$$v(t) = 3(t-3)(t-1) = 0$$

$$t = 1\ sec. \text{ and } t = 3\ sec.$$

d. The particle is moving in the positive direction when $v(t) > 0.$

$v(t) = 3(t-3)(t-1)$

so $v(t) = 0$ only at $t = 1,3.$

By the intermediate value theorem, since $v(t)$ is continuous everywhere, in order for it to change sign on an interval it must be equal to 0 someplace on that interval. Thus, $v(t)$ does not change sign on $t < 1,$ or $1 < t < 3,$ or $t > 3.$ So we only need to find the value of $v(t)$ at one point in each of those regions to determine if $v(t) > 0$ on that region.

For $t < 1$ we can choose $t = 0,$ and $v(0) = 9 > 0$ so

$v(t) > 0$ for $t < 1.$

For $1 < t < 3$ we can choose $t = 2,$ and

$v(2) = 3(2^2) - 12(2) + 9 = -3,$ so $v(t) < 0$ for $1 < t < 3.$

For $t > 3$ we can choose $t = 4$, and

$v(4) = 3(4^2) - 12(4) + 9 = 9 > 0$, so $v(t) > 0$ for $t > 3$.

Thus $v(t) > 0$ when $0 \leq t < 1$ or $t > 3$.

So the particle moves in the positive direction when $0 \leq t < 1$ or $t > 3$,

it's moving in the negative direction $(v(t) < 0)$ when $1 < t < 3$,

and it's at rest $(v(t) = 0)$, at $t = 1,3$.

e. The particle is moving in the positive direction for $0 \leq t < 1$.

So over that interval, the particle moves from $s(0) = 0$ to

$s(1) = 1^3 - 6(1)^2 + 9(1) = 4$. So it travels 4 feet.

The particle is moving in the negative direction for $1 < t < 3$.

So over that interval, the particle moves from $s(1) = 4$ to

$s(3) = 3^3 - 6(3)^2 + 9(3) = 0$. So it travels 4 feet in the negative direction.

The particle is moving in the positive direction for $3 < t \leq 5$.

So over that interval, the particle moves from $s(3) = 0$ to

$s(5) = 5^3 - 6(5)^2 + 9(5) = 20$. So it travels 20 feet in the positive direction.

Total distance travelled$= 4 + 4 + 20 = 28 feet$.

Average rate of change vs Instantaneous rate of change

For a function $y = f(x)$, we define the average rate of change over an interval $[a, a+h]$ to be:

Average rate of change $= \frac{f(a+h)-f(a)}{h}$.

The instantaneous rate of change of a function $y = f(x)$ at a point $x = a$ is defined to be:

Instantaneous rate of change $= \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$.

Ex. The daily output of a factory assembly line is estimated by:

$$P(t) = -t^2 + 40t \;\; units; \qquad 0 \le t \le 8 \; hours$$

a. What is the average hourly output in a day?

b. What is the instantaneous change in the output at $t = 3$ hours?

a. Average rate of change $= \frac{P(8)-P(0)}{8-0} = \frac{(-(8)^2+40(8))-(0)}{8-0}$

$$= \frac{(-64+320)-(0)}{8-0}$$

$$= \frac{256}{8} = 32 \; units/hr.$$

b. Instantaneous rate of change $= P'(t) = -2t + 40.$

At $t = 3$ we have: $P'(3) = -2(3) + 40$

$$= 34 \; units/hr \; .$$

Def. Let $C(x)$ be the cost function for producing x items. The **Average cost** to produce x items is $\bar{C}(x) = \frac{C(x)}{x}$. The **Marginal cost**, $C'(x)$, is the approximate cost of producing one more item after x items have been produced.

Ex. Suppose the cost in dollars of producing x items is given by

$$C(x) = -0.03x^2 + 60x + 100 \quad \text{for } 0 \le x \le 1000.$$

a. Determine the average cost function, $\bar{C}(x)$, and the marginal cost function, $C'(x)$.
b. Determine the average cost of producing 100 items. 200 items.
c. Determine the marginal cost after producing 100 items. 200 items.
d. Describe what $\bar{C}(100)$ and $C'(100)$ represent.

a. Average cost $= \bar{C}(x) = \frac{C(x)}{x} = \frac{-0.03x^2+60x+100}{x}$

Marginal cost $= C'(x) = -0.06x + 60.$

b. $\bar{C}(100) = \frac{C(100)}{100} = \frac{-0.03(100)^2+60(100)+100}{100}$

$= \frac{-300+6000+100}{100} = \$58/unit.$

$\bar{C}(200) = \frac{C(200)}{200} = \frac{-0.03(200)^2+60(200)+100}{200}$

$= \frac{-1200+12000+100}{200} = \$54.50/unit.$

c. $C'(100) = -0.06(100) + 60 = \$54/unit.$

$C'(200) = -0.06(200) + 60 = \$48/unit.$

d. $\bar{C}(100) =$average cost to produce 100 units

$C'(100) =$approximate cost to produce the 101st unit.