Derivatives of Trigonometric Functions

To determine the derivatives of trigonometric functions we are going to need the following 2 facts:

$$\lim_{x \to 0} \frac{sinx}{x} = 1 \qquad \text{and} \qquad \lim_{x \to 0} \frac{cosx - 1}{x} = 0.$$

Ex. Evaluate the following limits

a. $\lim_{x \to 0} \frac{sin6x}{x}$ b. $\lim_{x \to 0} \frac{sin3x}{sin8x}$

c. $\lim_{x \to 0} \frac{x}{sin4x}$ d. $\lim_{x \to 1} \frac{\tan(x-1)}{x^2-1}$.

a. $\lim_{x \to 0} \frac{sin6x}{x} = \lim_{x \to 0} (\frac{sin6x}{6x})6$

let $u = 6x$; $\lim_{x \to 0} (\frac{sin6x}{6x})6 = \lim_{u \to 0} (\frac{sinu}{u})6 = 6.$

b. $\lim_{x \to 0} \frac{sin3x}{sin8x} = \lim_{x \to 0} \frac{(\frac{sin3x}{3x})3}{(\frac{sin8x}{8x})8} = \frac{3}{8}$.

c. $\lim_{x \to 0} \frac{x}{sin4x} = \lim_{x \to 0} \left(\frac{4x}{sin4x}\right)(\frac{1}{4}) = \lim_{x \to 0} \left(\frac{1}{\frac{sin4x}{4x}}\right)(\frac{1}{4}) = \frac{1}{4}$.

d. $\lim_{x\to1}\frac{\tan(x-1)}{x^2-1}=\lim_{x\to1}\frac{\tan(x-1)}{(x-1)(x+1)}=\lim_{x\to1}\frac{\frac{\sin(x-1)}{\cos(x-1)}}{(x-1)(x+1)}$

$$=\lim_{x\to1}\left[\frac{\sin(x-1)}{(x-1)}\right]\left[\frac{1}{(x+1)\cos(x-1)}\right]; \quad \text{Let } u=x-1$$

$$=\lim_{u\to0}\left[\frac{\sin(u)}{(u)}\right]\left[\frac{1}{(u+2)\cos(u)}\right]=1\left(\frac{1}{2(1)}\right)=\frac{1}{2}.$$

<u>Derivatives of $sinx$ and $cosx$</u>

Let $f(x)=sinx$, recall that

$$\sin(x+h)=(sinx)(\cos(h))+(\sin(h))(\cos x).$$

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{\sin(x+h)-sinx}{h}$$

$$=\lim_{h\to0}\frac{(sinx)(\cos(h))+(\sin(h))(\cos x)-sinx}{h}$$

$$=\lim_{h\to0}\frac{\text{sinx}(\cos(\text{h})-1)+(\sin(\text{h}))(\text{cosx})}{h}$$

$$=\lim_{h\to0}\left[\frac{(\text{sinx})(\cos(\text{h})-1)}{h}\right]+\lim_{h\to0}\frac{(\sin(\text{h}))(\text{cosx})}{h}.$$

on the first limit we use $\lim_{x\to0}\frac{cosx-1}{x}=0$ and on the second $\lim_{x\to0}\frac{sinx}{x}=1.$

$$f'(x)=0+cosx=cosx.$$

$$\frac{d}{dx}(\boldsymbol{sinx}) = \boldsymbol{cosx}$$

$$\frac{d}{dx}(\boldsymbol{cosx}) = -\boldsymbol{sinx}\ .$$

The derivative of $cosx$ can be gotten in a similar way to $sinx$ but using the formula for $cos(x+h)$ instead of $sin(x+h)$.

Notice that for $f(x) = sinx$ or $g(x) = cosx$ we have

$f'(x) = cosx$	$g'(x) = -sinx$
$f''(x) = -sinx$	$g''(x) = -cosx$
$f'''(x) = -cosx$	$g'''(x) = sinx$
$f^{(4)}(x) = sinx$	$g^{(4)}(x) = cosx$

So the derivatives of $sinx$ and $cosx$ cycle. This becomes useful when one studies Taylor series (usually 2nd semester Calc.).

So if we want to know $f^{(42)}(x)$ and $g^{(42)}(x)$, all we have to do is divide the 42 by 4 and take the remainder (2). Thus

$f^{(42)}(x) = f''(x) = -sinx$ and $g^{(42)}(x) = g''(x) = -cosx$.

This is a special feature of $f(x) = sinx$ and $g(x) = cosx$.

Ex. If $f(x) = sinx$ and $g(x) = cosx$ find $f^{(75)}(x)$ and $g^{(75)}(x)$.

$\frac{75}{4} = 18$ with a remainder of 3. Thus we have:

$$f^{(75)}(x) = f^{(3)}(x) = -cosx$$

$$g^{(75)}(x) = g^{(3)}(x) = sinx.$$

Ex. $f(x) = cosx$, find $f'(\frac{\pi}{6})$, $f''(\frac{\pi}{6})$.

$$f'(x) = -sinx \implies f'\left(\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

$$f''(x) = -cosx \implies f''\left(\frac{\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}.$$

Ex. $g(x) = x^2 cosx + 4sinx$ find $g'(x)$.

Remember, we need to use the product rule on $x^2 cosx$:

$$g'(x) = x^2 \frac{d}{dx}(cosx) + (cosx)\frac{d}{dx}(x^2) + 4cosx$$

$$= x^2(-sinx) + (cosx)(2x) + 4cosx$$

$$= (2x + 4)cosx - x^2 sinx.$$

Ex. $f(x) = 4(sinx)(cosx)$, find $f'(x)$.

Once again, we need to use the product rule:

$$f'(x) = 4[sinx\left(\frac{d}{dx}(cosx)\right) + cosx\left(\frac{d}{dx}(sinx)\right)]$$

$$= 4[sinx(-sinx) + cosx(cosx)]$$

$$= 4(\cos^2 x - \sin^2 x).$$

Ex. $h(t) = \frac{sint}{t^2+1}$, find $h'(t)$.

In this example we need to use the quotient rule:

$$h'(t) = \frac{(t^2+1)\frac{d}{dt}(sint) - (sint)\frac{d}{dt}(t^2+1)}{(t^2+1)^2}$$

$$= \frac{(t^2+1)(cost) - (sint)(2t)}{(t^2+1)^2} .$$

Ex. $f(x) = \frac{cosx}{1-cosx}$ find $f'(x)$.

Again we need to use the quotient rule:

$$f'(x) = \frac{(1-cosx)\frac{d}{dx}(cosx) - (cosx)\frac{d}{dx}(1-cosx)}{(1-cosx)^2}$$

$$= \frac{(1-cosx)(-sinx) - (cosx)(sinx)}{(1-cosx)^2}$$

$$= \frac{-sinx + (cosx)(sinx) - (cosx)(sinx)}{(1-cosx)^2}$$

$$= \frac{-sinx}{(1-cosx)^2} .$$

Since the other 4 trig functions are defined in terms of $sinx$ and $cosx$, we can use our differentiation rules to calculate their derivatives. For example:

$\frac{d}{dx}(tanx) = \frac{d}{dx}(\frac{sinx}{cosx})$; Now use the quotient rule.

$$= \frac{cosx\left(\frac{d}{dx}(sinx)\right) - sinx(\frac{d}{dx}(cosx))}{(cosx)^2}$$

$$= \frac{cosx(cosx) - sinx(-sinx)}{(cosx)^2} = \frac{(cosx)^2 + (sinx)^2}{(cosx)^2} = \frac{1}{(cosx)^2}$$

$$\frac{d}{dx}(tanx) = sec^2x.$$

The derivatives of $cscx$, $secx$, and $cotx$ can all be gotten from defining those functions in terms of $sinx$ and/or $cosx$ and using the quotient rule.

Derivatives of Trig Functions (know these cold!):

$\boldsymbol{\frac{d}{dx}(sinx) = cosx}$	$\boldsymbol{\frac{d}{dx}(cscx) = -cscx(cotx)}$
$\boldsymbol{\frac{d}{dx}(cosx) = -sinx}$	$\boldsymbol{\frac{d}{dx}(secx) = secx(tanx)}$
$\boldsymbol{\frac{d}{dx}(tanx) = sec^2x}$	$\boldsymbol{\frac{d}{dx}(cotx) = -csc^2x}$

Notice that the derivatives of all of the “co” functions $(cosx, cscx, cotx)$ have minus signs in their derivative formulas while the others don’t.

Ex. Let $f(x) = xsinx + (cscx)(cotx)$. Find $f'(x)$.

We need to use the product rule for each of the 2 terms:

$$f'(x) = x\frac{d}{dx}(sinx) + sinx\left(\frac{d}{dx}(x)\right) + cscx(\frac{d}{dx}cotx) + cotx(\frac{d}{dx}cscx)$$

$$= xcosx + sinx + cscx(-csc^2x) + cotx(-(cscx)(cotx))$$

$$= xcosx + sinx - csc^3x - (cscx)(cot^2x)$$

Ex. Let $g(t) = \frac{2costsint}{\sqrt{t}}$, find $g'(t)$.

Here we have a product of 2 functions in the numerator in addition to a quotient of functions. So we will need the quotient rule and the product rule. Also, remember we earlier found that $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$.

$$g'(t) = \frac{\sqrt{t}\frac{d}{dt}(2costsint) - 2cost(sint)\frac{d}{dt}(\sqrt{t})}{(\sqrt{t})^2}$$

$$= \frac{\sqrt{t}\left(2cost(\frac{d}{dt}(sint)) + sint\left(\frac{d}{dt}(2cost)\right)\right) - 2cost(sint)(\frac{1}{2\sqrt{t}})}{t}$$

$$= \frac{\sqrt{t}\left(2cos^2t - 2sin^2t\right) - \frac{cost(sint)}{\sqrt{t}}}{t} \qquad \text{(now multiple by } \frac{\sqrt{t}}{\sqrt{t}}\text{)}$$

$$= \frac{2t\left(cos^2t - sin^2t\right) - cost(sint)}{t^{\frac{3}{2}}}$$

Ex. Find $\frac{dy}{dx}$ when $y = \frac{(secx)(tanx)}{x^2}$.

Start with the quotient rule:

$$\frac{dy}{dx} = \frac{x^2(\frac{d}{dx}((secx)(tanx)) - (secx)(tanx)\frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{x^2\left[(secx)\frac{d}{dx}(tanx)+(tanx)\frac{d}{dx}(secx)\right]-(secx)(tanx)(2x)}{x^4}$$

$$= \frac{x^2[(secx)(\sec^2 x)+(tanx)(secx)(tanx)]-2x(secx)(tanx)}{x^4}$$

$$= \frac{x[xsec^3x+x(\tan^2 x)(secx)-2(secx)(tanx)]}{x^4}$$

$$= \frac{xsec^3x+x(\tan^2 x)(secx)-2(secx)(tanx)}{x^3}.$$

Ex. For what values of x does the graph of $f(x) = 6cosx + 3x$ have a horizontal tangent line?

The graph of $f(x) = 6cosx + 3x$ has a horizontal tangent line when $f'(x) = 0$.

$$f'(x) = -6sinx + 3 = 0 \Longrightarrow \quad 3 = 6sinx \Longrightarrow \quad sinx = \frac{1}{2}.$$

So the graph of $f(x) = 6cosx + 3x$ has a horizontal tangent line at all points where $sinx = \frac{1}{2}$.

$x = \frac{\pi}{6} + 2n\pi$; where n is an integer or

$x = \frac{5\pi}{6} + 2n\pi$; where n is an integer.