The Product and Quotient Rules

We want to develop rules that allow us to calculate $\frac{d}{dx}(f(x)g(x))$ and $\frac{d}{dx}(\frac{f(x)}{g(x)})$.

Unfortunately, $\frac{d}{dx}(f(x)g(x)) \neq \frac{d}{dx}(f(x))\frac{d}{dx}(g(x))$

$$\frac{d}{dx}(\frac{f(x)}{g(x)}) \neq \frac{d}{dx}(f(x))/\frac{d}{dx}(g(x)).$$

Derivative Rule 5: The Product Rule

If $f(x)$ and $g(x)$ are differentiable at x then

$$\frac{d}{dx}\big(f(x)g(x)\big) = f(x)g'(x) + g(x)f'(x).$$

Ex. Find $\frac{d}{dx}[(2x^5 - 3)(4x^3 + 1)]$.

Here we can let: $f(x) = 2x^5 - 3,$ so $f'(x) = 10x^4$

$g(x) = 4x^3 + 1,$ so $g'(x) = 12x^2.$

$$\frac{d}{dx}\big(f(x)g(x)\big) = f(x)g'(x) + g(x)f'(x).$$

$$\frac{d}{dx}\Big((2x^5 - 3)(4x^3 + 1)\Big) = (2x^5 - 3)(12x^2) + (4x^3 + 1)(10x^4)$$

$$= 24x^7 - 36x^2 + 40x^7 + 10x^4$$

$$= 64x^7 + 10x^4 - 36x^2.$$

Ex. Find $\frac{d}{dx}((3x^4 - 2x)(4x^2 + 3))$

Let: $f(x) = 3x^4 - 2x,$ so $f'(x) = 12x^3 - 2$

$g(x) = 4x^2 + 3,$ so $g'(x) = 8x.$

$$\frac{d}{dx}\big(f(x)g(x)\big) = f(x)g'(x) + g(x)f'(x).$$

$$\frac{d}{dx}\big(3x^4 - 2x)(4x^2 + 3)\big) = (3x^4 - 2x)(8x) + (4x^2 + 3)(12x^3 - 2)$$

$$= (24x^5 - 16x^2) + (48x^5 + 36x^3 - 8x^2 - 6)$$

$$= 72x^5 + 36x^3 - 24x^2 - 6.$$

Derivative Rule 6: The Quotient Rule

If $f(x)$ and $g(x)$ are differentiable at x and $g(x) \neq 0$ then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

Ex. Find $\frac{d}{dx}\left(\frac{2x^5 - 3}{4x^3 + 1}\right)$.

As in our first example, let: $f(x) = 2x^5 - 3,$ so $f'(x) = 10x^4$

$g(x) = 4x^3 + 1,$ so $g'(x) = 12x^2$.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$$

$$\frac{d}{dx}\left(\frac{2x^5-3}{4x^3+1}\right) = \frac{\left(4x^3+1\right)(10x^4)-(2x^5-3)(12x^2)}{(4x^3+1)^2}$$

$$= \frac{40x^7+10x^4-24x^7+36x^2}{(4x^3+1)^2}$$

$$= \frac{16x^7+10x^4+36x^2}{(4x^3+1)^2}\,.$$

Ex. Find $\frac{d}{dx}\left(\frac{2x-1}{3x^2+1}\right)$

Let: $f(x) = 2x - 1$ so $f'(x) = 2$
$g(x) = 3x^2 + 1$ so $g'(x) = 6x$.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$$

$$\frac{d}{dx}\left(\frac{2x-1}{3x^2+1}\right) = \frac{\left(3x^2+1\right)(2)-(2x-1)(6x)}{(3x^2+1)^2}$$

$$= \frac{\left(6x^2+2\right)-(12x^2-6x)}{(3x^2+1)^2}$$

$$= \frac{6x^2+2-12x^2+6x}{(3x^2+1)^2}$$

$$= \frac{-6x^2+6x+2}{(3x^2+1)^2}\,.$$

Ex. Find an equation of a tangent line to the graph of $f(x) = \frac{2x^2+1}{x^2-2}$ at the point $(1, -3)$.

$$f'(x) = \frac{(x^2-2)\frac{d}{dx}(2x^2+1)-(2x^2+1)\frac{d}{dx}(x^2-2)}{(x^2-2)^2}$$
$$= \frac{(x^2-2)(4x)-(2x^2+1)(2x)}{(x^2-2)^2}.$$

For this problem we only need to know $f'(1)$ so we can just plug 1 into the calculation for $f'(x)$.

$$f'(1) = \frac{(1^2-2)(4(1))-(2(1^2)+1)(2(1))}{(1^2-2)^2}$$
$$= \frac{(-1)4-(3)(2)}{(-1)^2} = \frac{-4-6}{1} = -10.$$

So the slope of the tangent line to the graph of $f(x) = \frac{2x^2+1}{x^2-2}$ at the point $(1, -3)$ is -10.

Eq. of tangent line: $y + 3 = -10(x - 1)$.

Derivative Rule 2 (revisited): The Power Rule

For ANY integer n, $\frac{d}{dx}(x^n) = nx^{n-1}$.

Since we know this rule for $n \geq 0$, we only need to show it for $n < 0$.

Suppose $n = -m,\ m > 0$.

Since $x^{-m} = \frac{1}{x^m}$ by the quotient rule we have:

$$\frac{d}{dx}(x^n) = \frac{d}{dx}(x^{-m}) = \frac{d}{dx}\left(\frac{1}{x^m}\right) = \frac{x^m \frac{d}{dx}(1) - 1(\frac{d}{dx}(x^m))}{(x^m)^2}$$

$$= \frac{x^m(0) - mx^{m-1}}{x^{2m}}$$

$$= -mx^{-m-1} = nx^{n-1}.$$

You might be tempted to use the quotient rule when taking the derivative of any quotient, however, if you can turn the quotient into powers of x, it's easier to use the power rule. For example:

Ex. Find the derivatives of the following functions

a. $f(x) = \frac{5}{x^7}$
b. $g(t) = \frac{4t^8 - 2t^2}{t^3}$

a. $f(x) = \frac{5}{x^7} = 5x^{-7}$, so $f'(x) = -35x^{-8}$

b. $g(t) = \frac{4t^8 - 2t^2}{t^3} = 4\frac{t^8}{t^3} - 2\frac{t^2}{t^3} = 4t^5 - 2t^{-1}$

So $g'(t) = 20t^4 + 2t^{-2}$.

Ex. Find the derivative of $f(x) = \frac{4}{x^3} + \frac{2x^6 - 3x}{x^4}$.

$$f(x) = \frac{4}{x^3} + \frac{2x^6-3x}{x^4} = \frac{4}{x^3} + 2\left(\frac{x^6}{x^4}\right) - 3\left(\frac{x}{x^4}\right)$$
$$= 4x^{-3} + 2x^2 - 3x^{-3}$$
$$= x^{-3} + 2x^2.$$

$$f'(x) = -3x^{-4} + 4x = -\frac{3}{x^4} + 4x.$$

Ex. Find an equation of the tangent line to the graph of $f(x) = \frac{x+x^{-1}}{2x^2-1}$ at the point $(1,2)$.

Let's do this 2 ways.

#1. Simplify the function before taking the derivative.

$$f(x) = \frac{x+x^{-1}}{2x^2-1} = \frac{x+\frac{1}{x}}{2x^2-1} = \left(\frac{x}{x}\right)\left(\frac{x+\frac{1}{x}}{2x^2-1}\right) = \frac{x^2+1}{2x^3-x}$$

Now use the quotient rule:

$$f'(x) = \frac{(2x^3-x)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(2x^3-x)}{(2x^3-x)^2}$$
$$= \frac{(2x^3-x)(2x) - (x^2+1)(6x^2-1)}{(2x^3-x)^2}.$$

Since we are just finding a tangent line at $x = 1$ we only need $f'(1)$

$$f'(1) = \frac{\left(2(1^3)-1\right)\left(2(1)\right) - (1^2+1)\left(6(1^2)-1\right)}{(2(1^3)-1)^2}$$
$$= \frac{2-(2)(5)}{1} = -8.$$

Eq. of tangent line at $(1,2)$: $\qquad y - 2 = -8(x-1).$

#2. Don't simplify the function, just apply the quotient rule.

$$f'(x) = \frac{(2x^2-1)\frac{d}{dx}(x+x^{-1})-(x+x^{-1})\frac{d}{dx}(2x^2-1)}{(2x^2-1)^2}$$

$$= \frac{(2x^2-1)(1-x^{-2})-(x+x^{-1})4x}{(2x^2-1)^2}$$

Now plug in $x = 1$:

$$f'(1) = \frac{(2(1^2)-1)(1-1^{-2})-(1+1^{-2})(4(1))}{(2(1^2)-1)^2}$$

$$= \frac{1(0)-(2)(4)}{1} = -8,$$

So once again we have:
Eq. of tangent line at $(1,2)$: $y - 2 = -8(x - 1).$