Some Differentiation Rules

Calculating a derivative from a limit definition of a derivative can be very difficult. Fortunately, we can develop some rules to make this calculation easier.

Derivative Rule 1: Constant Rule

$\frac{d}{dx}(c) = 0$ where c is a constant.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = 0.$$

Derivative Rule 2: The Power Rule

$\frac{d}{dx}(x^n) = nx^{n-1}$ if n is a non-negative integer.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x) = \lim_{h\to 0}\frac{(x+h)^n-x^n}{h}$$

$$= \lim_{h\to 0}\frac{x^n+nhx^{n-1}+\cdots+h^n-x^n}{h}$$

$$= \lim_{h\to 0}\frac{nhx^{n-1}+\cdots+h^n}{h}$$

$$= \lim_{h\to 0}\frac{h(nx^{n-1}+\cdots+h^{n-1})}{h}$$

$$= \lim_{h\to 0}(nx^{n-1}+\cdots+h^{n-1}) = nx^{n-1}.$$

Ex. find $f'(x)$ for the following functions

a. $f(x) = \pi^4$
b. $f(x) = x^{100}$
c. $f(x) = x.$

a. $f'(x) = 0,$ by the constant rule.

b. $f'(x) = 100x^{99},$ by the power rule.

c. $f'(x) = 1,$ by the power rule $(x = x^1)$.

Derivative Rule 3: The constant multiple rule

$$\frac{d}{dx}(cf(x)) = cf'(x).$$

If $g(x) = cf(x)$ then $g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$

$$= \lim_{h\to 0} \frac{cf(x+h)-cf(x)}{h}$$

$$= c \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = cf'(x).$$

Ex. find $g'(x)$ for $g(x) = -\frac{5x^8}{3}$.

$$g(x) = -\frac{5}{3}x^8, \text{ so } g'(x) = \left(-\frac{5}{3}\right)(8x^7) = -\frac{40}{3}x^7.$$

Derivative Rule 4: The Sum/Difference Rule

$\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$

$\frac{d}{dx}(f(x)-g(x)) = f'(x)-g'(x)$.

$$\begin{aligned}\frac{d}{dx}(f(x)+g(x)) &= \lim_{h\to 0}\frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h}\\ &= \lim_{h\to 0}\frac{(f(x+h)-f(x))+(g(x+h)-g(x))}{h}\\ &= \lim_{h\to 0}\frac{(f(x+h)-f(x))}{h}+\lim_{h\to 0}\frac{\left(g(x+h)-g(x)\right)}{h}\\ &= f'(x)+g'(x).\end{aligned}$$

Notice that the sum rule works for any finite number of functions.

$\frac{d}{dx}(f_1(x)+f_2(x)+\cdots+f_n(x)) = f_1{}'(x)+f_2{}'(x)\ldots+f_n{}'(x)$.

The same proof for the Sum Rule can also be used for the Difference Rule

$\frac{d}{dx}(f(x)-g(x)) = f'(x)-g'(x)$.

We can now find the derivative of any polynomial

Ex. Let $f(x) = 3x^{12} - 5x^8 + 2x^2 - x + 7$. Find $f'(x)$.

$$f'(x) = \frac{d}{dx}(3x^{12} - 5x^8 + 2x^2 - x + 7)$$

$$= \frac{d}{dx}(3x^{12}) - \frac{d}{dx}(5x^8) + \frac{d}{dx}(2x^2) - \frac{d}{dx}(x) + \frac{d}{dx}(7) \quad \text{(Sum/ Diff. rules)}$$

$$= 3\frac{d}{dx}(x^{12}) - 5\frac{d}{dx}(x^8) + 2\frac{d}{dx}(x^2) - \frac{d}{dx}(x) + \frac{d}{dx}(7) \quad \text{(Const. Mult. Rule)}$$

$$= 3(12x^{11}) - 5(8x^7) + 2(2x) - 1 + 0 \quad \text{(Power Rule)}$$

$$= 36x^{11} - 40x^7 + 4x - 1.$$

Ex. Let $f(x) = x^3 + 3x^2 - 9x + 4$. For what values of x does the slope of the tangent line to the graph of $y = f(x)$ equal 0?

$$f'(x) = \frac{d}{dx}(x^3 + 3x^2 - 9x + 4)$$

$$= \frac{d}{dx}(x^3) + \frac{d}{dx}(3x^2) - \frac{d}{dx}(9x) + \frac{d}{dx}(4)$$

$$= \frac{d}{dx}(x^3) + 3\frac{d}{dx}(x^2) - 9\frac{d}{dx}(x) + \frac{d}{dx}(4)$$

$$= 3x^2 + 6x - 9 = 0$$

$$3(x^2 + 2x - 3) = 0$$

$$3(x + 3)(x - 1) = 0$$

$$x = -3, 1.$$

So $f'(-3) = 0, \quad f'(1) = 0.$

Ex. Find the Derivatives:

a. $f(x) = (3x^3 - x)(2x^2 + 1)$

b. $g(x) = \frac{4x^8 - 2x^6}{x^3}$.

a. $f(x) = (3x^3 - x)(2x^2 + 1) = 6x^5 + 3x^3 - 2x^3 - x$

$$= 6x^5 + x^3 - x$$

$$f'(x) = \frac{d}{dx}(6x^5) + \frac{d}{dx}(x^3) - \frac{d}{dx}(x)$$

$$= 6\frac{d}{dx}(x^5) + \frac{d}{dx}(x^3) - \frac{d}{dx}(x)$$

$$= 6(5x^4) + 3x^2 - 1 = 30x^4 + 3x^2 - 1.$$

b. $g(x) = \frac{4x^8 - 2x^6}{x^3} = \frac{4x^8}{x^3} - \frac{2x^6}{x^3} = 4x^5 - 2x^3$

$$g'(x) = \frac{d}{dx}(4x^5) - \frac{d}{dx}(2x^3)$$

$$= 4\frac{d}{dx}(x^5) - 2\frac{d}{dx}(x^3)$$

$$= 4(5x^4) - 2(3x^2) = 20x^4 - 6x^2.$$

Higher Order Derivatives

Given a function $f(x)$, its derivative, $f'(x)$ is also a function. So we can try to compute the derivative of $f'(x)$ which we denote $f''(x)$. This is called the second derivative of $f(x)$.

We can continue this process and try to take the nth derivative of the function $f(x)$, which we denote $f^{(n)}(x)$. Other common notations for the nth derivative of $f(x)$ are: $\frac{d^n y}{dx^n}$, $\frac{d^n f}{dx^n}$, and $y^{(n)}$.

Ex. Find the third derivative of each of the following functions.

a. $f(x) = 3x^{12} - 5x^8 + 2x^2 - x + 7$

b. $y = t^6 + 3t^2 + 2t - 4$

a. From an earlier example:

$$f'(x) = 36x^{11} - 40x^7 + 4x - 1,$$

so $f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}(36x^{11} - 40x^7 + 4x - 1)$

$$= 396x^{10} - 280x^6 + 4$$

$$f'''(x) = \frac{d}{dx}\left(f''(x)\right) = \frac{d}{dx}(396x^{10} - 280x^6 + 4)$$

$$= 396(10x^9) - 280(6x^5)$$

$$= 3960x^9 - 1680x^5.$$

b. $\frac{dy}{dt} = \frac{d}{dt}(t^6 + 3t^2 + 2t - 4)$

$$= 6t^5 + 6t + 2.$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right)$$

$$= \frac{d}{dt}(6t^5 + 6t + 2)$$

$$= 30t^4 + 6.$$

$$\frac{d^3y}{dt^3} = \frac{d}{dt}\left(\frac{d^2y}{dt^2}\right)$$

$$= \frac{d}{dt}(30t^4 + 6)$$

$$= 120t^3.$$