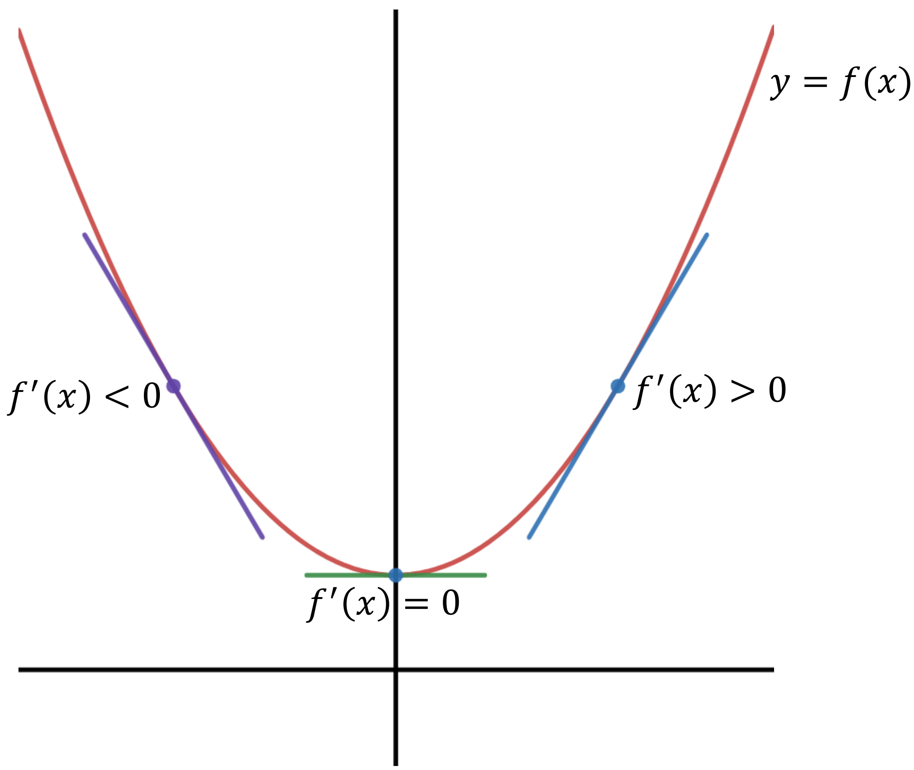
Algebra Review for Calculus

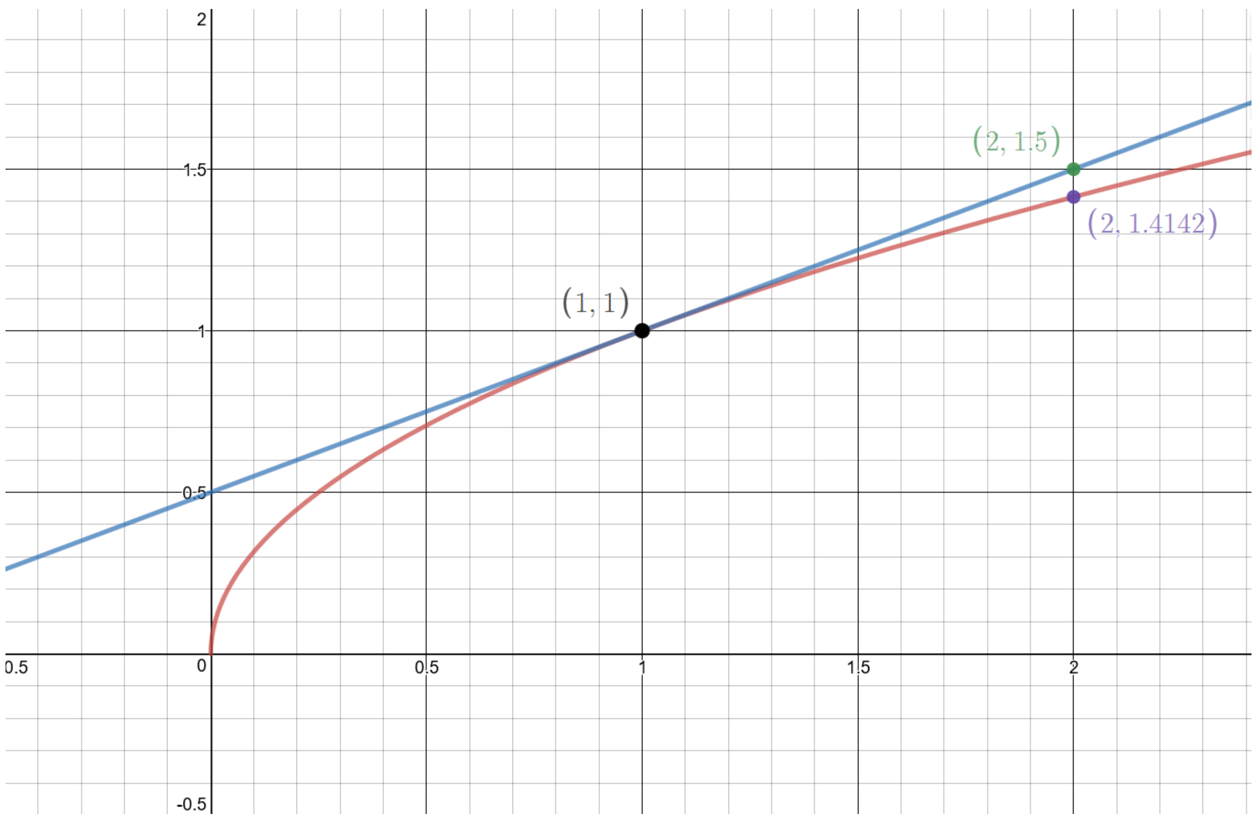
0. Introduction

You will spend a fair amount of time in Calculus finding "Derivatives" of functions. A "derivative" of a function $f(x)$ is another function, $f'(x)$, that gives the slope of the tangent line to the graph of $f(x)$ for each value of x.



Derivatives can tell us a lot about a function. For example, when $f'(x) > 0$ a function is increasing (its graph is going up) and when $f'(x) < 0$ a function is decreasing. Also, when a function hits certain types of maximum and minimum values $f'(x) = 0$.

Since the equation of a line is very simple, the tangent line to a graph can also be used to approximate the value of a complex function. For example, we can use the tangent line to the function $f(x) = \sqrt{x}$ at the point $(1,1)$ to approximate the value of $\sqrt{2}$.



The biggest stumbling block to success in Calculus tends not to be the new concepts one learns in Calculus, but rather the Algebra and Trigonometry that is required to do the Calculus. It is already evident from what we've done so far that one has to be able to:

1. Find equations of (tangent) lines
2. Solve equations (eg, when is $f'(x) = 0$?)
3. Solve inequalities (eg, when is $f'(x) > 0$?).

One also needs to be comfortable working with the various types of functions whose derivative you will calculate (eg, polynomials, exponential functions, log functions, trig functions, etc.). Thus one's Algebra skills must be strong to do Calculus.

1. <u>Equations of Lines:</u> The Point-Slope Method

m=slope of line, (x_1, y_1) is a point on the line then the equation of the line is:

$$y - y_1 = m(x - x_1).$$

Ex. Find an equation of a line through the point (-2,4) whose slope is -3.

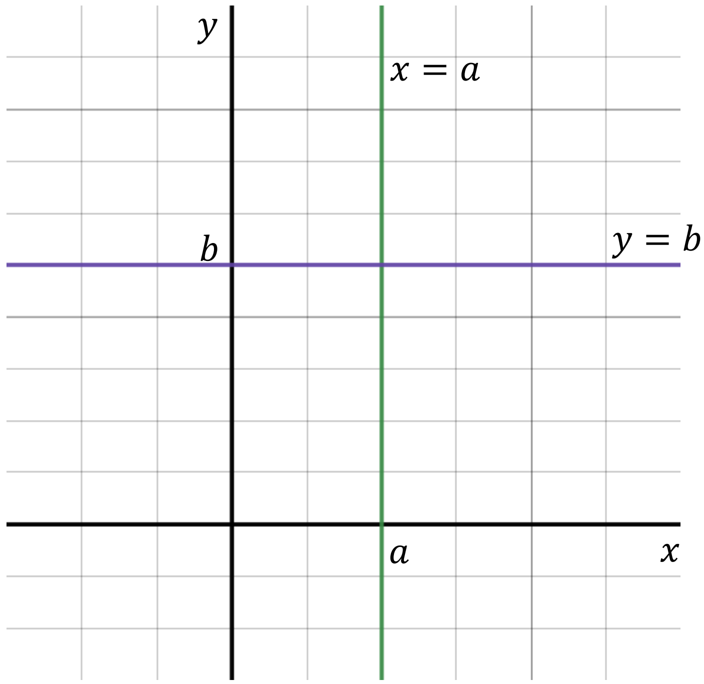
$$(x_1, y_1) = (-2,4), \qquad m = -3$$

An equation of the line is: $y - 4 = -3(x - (-2)) = -3(x + 2)$.

Horizontal and Vertical lines:

The equation of a horizontal line through the point (a, b) is $y = b$

The equation of a vertical line through the point (a, b) is $x = a$.



Ex. Find the equations of the horizontal and vertical lines through the point $(-3,4)$.

Horizontal line: $y = 4$

Vertical line: $x = -3$

2. Functions

Evaluating functions:

Ex. $f(x) = x^2 - x$, find $f(-3)$, $f(x+2)$, $\frac{f(x+h)-f(x)}{h}$

$$f(-3) = (-3)^2 - (-3) = 9 + 3 = 12$$

$$f(x+2) = (x+2)^2 - (x+2) = (x^2 + 4x + 4) - x - 2 = x^2 + 3x + 2$$

$$\frac{f(x+h)-f(x)}{h} = \frac{[(x+h)^2-(x+h)]-(x^2-x)}{h}$$

$$= \frac{\left[\left(x^2+2hx+h^2\right)-x-h\right]-x^2+x}{h}$$

$$= \frac{x^2+2hx+h^2-x-h-x^2+x}{h}$$

$$= \frac{2hx+h^2-h}{h}$$

$$= \frac{h(2x+h-1)}{h}$$

$$= 2x + h - 1$$

Composition of Functions:

Ex. $f(x) = x^2 - x$, $g(x) = x - 2$, Find $f(g(x))$ and $g(f(x))$.

$$f\big(g(x)\big) = f(x-2) = (x-2)^2 - (x-2)$$
$$= (x^2 - 4x + 4) - x + 2$$
$$= x^2 - 5x + 6$$

$$g(f(x)) = g(x^2 - x) = (x^2 - x) - 2$$
$$= x^2 - x - 2.$$

3. <u>Solving Equations</u>

Quadratic Equations: Equations that can be put in the form: $ax^2 + bx + c = 0$

Ex. $2x^2 = 6 - x$; Subtract 6 and add x to both sides.

$$2x^2 + x - 6 = 0.$$

To solve quadratic equations, we get all non-zero terms on one side and then either:

1. Factor the equation (if you can). Note: if $b^2 - 4ac$ is a perfect square it can be factored.

2. Use the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$; if $b^2 - 4ac < 0$, there are no real roots.

Ex. Solve $x^2 - 2x = 15$.

First get all non-zero terms on one side of the equation (it doesn't matter which side). Subtract 15 from both sides.

$x^2 - 2x - 15 = 0$

Try to factor first. If that doesn't work then use the quadratic formula.

$(x-5)(x+3) = 0$

Now set each factor equal to 0:

$x - 5 = 0, \quad x + 3 = 0$.

$x = 5$ or $x = -3$.

Note: If you have a negative number in front of the x^2 term, you can always make it positive by multiplying the equation by -1. This is useful to do since it's difficult to factor with negative numbers in front of the x^2 term.

Ex. Solve $x^2 = 25$.

Solution 1: subtract 25 from both sides and factor.

$x^2 - 25 = 0$

$(x-5)(x+5) = 0$

$x - 5 = 0, \quad x + 5 = 0$

$x = 5 \quad or \quad x = -5$

Solution 2: Take square roots of each side. **BE CAREFUL WHEN YOU DO THAT!!!!**

There are 2 square roots, a positive one and a negative one.

$x = +5, \ -5$

Don't forget the negative square root.

Ex. Solve $2x^2 = 2 - 3x$.

First get all non-zero terms on one side. Subtract 2 and add $3x$ to both sides.

$2x^2 + 3x - 2 = 0$.

I would suggest trying to factor the expression. If you can't do it in 30 seconds use the quadratic formula.

This is factorable (because $b^2 - 4ac$ is a perfect square, 25 in this case), but let's say you can't do it in 30 seconds. So let's use the quadratic formula.

$a = 2$, $b = 3$, $c = -2$ (be careful that you have the correct signs):

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(2)(-2)}}{2(2)}$$

$$x = \frac{-3 \pm \sqrt{9+16}}{4} = \frac{-3 \pm \sqrt{25}}{4} = \frac{-3 \pm 5}{4}$$

$x = \frac{-3+5}{4} = 1/2$ or $x = \frac{-3-5}{4} = -2$.

Solving Non-Quadratic Equations:

The following types of equations frequently show up when you do optimization (max/min) problems in Calculus.

Ex. Solve $4 - \frac{64}{x^2} = 0.$

The problem is the x^2 in the denominator. To get rid of that we multiply both sides of the equation by x^2.

$x^2(4 - \frac{64}{x^2}) = 0(x^2)$

$4x^2 - 64 = 0$; divide both sides of the equation by 4

$x^2 - 16 = 0$; Now either factor or add 16 to both sides and take 2 square roots

$(x - 4)(x + 4) = 0$

$x = 4$ or $x = -4.$

Ex. Solve $2x - \frac{54}{x^2} = 0.$

Again, the problem is the x^2 in the denominator. We multiply both sides of the equation by x^2.

$$x^2(2x - \frac{54}{x^2}) = 0(x^2)$$

$2x^3 - 54 = 0$; Divide the equation by 2

$x^3 - 27 = 0$; Add 27 to both sides

$$x^3 = 27$$

$$x = 3 .$$

4. <u>Simplifying Fractions</u>

Simplifying fractions often comes up when you deal with rational functions, i.e. one polynomial divided by another polynomial ($f(x) = \frac{P(x)}{Q(x)}$) . Usually, the key to simplifying fractions is factoring the numerator and the denominator and then cancelling common factors.

Ex. Simplify $\frac{x^2+8x+15}{x^2+x-6}$

$$\frac{x^2+8x+15}{x^2+x-6} = \frac{(x+3)(x+5)}{(x+3)(x-2)} = \frac{(x+5)}{(x-2)} .$$

Ex. When finding the derivative of $\frac{2x^2+1}{(x+1)^4}$, one has to simplify the following (messy) fraction: $\frac{(x+1)^4(4x)-(2x^2+1)(4)(x+1)^3}{(x+1)^8}$.

Notice that in the numerator we can factor out $(x+1)^3$ of both terms.

$= \frac{(x+1)^3[(x+1)(4x)-(2x^2+1)(4)]}{(x+1)^8}$ Now we can cancel 3 powers of $(x+1)$ in the numerator and the denominator.

$= \frac{[(x+1)(4x)-(2x^2+1)(4)]}{(x+1)^5}$. Now multiply out the top and recombine.

$= \frac{4x^2+4x-8x^2-4}{(x+1)^5}$

$= \frac{-4x^2+4x-4}{(x+1)^5}$

$= \frac{-4(x^2-x+1)}{(x+1)^5}$

Adding and Subtracting fractions:

When finding the derivative of a rational function from the definition of a derivative, you generally have to deal with complex fractions. This often involves subtracting fractions in the numerator of a complex fraction. To subtract fractions, we need to find a common denominator. Often this is just the product of the denominators of the fractions.

Ex. $$\frac{2}{x+5} - \frac{3}{x-2} = \frac{2(x-2)}{(x+5)(x-2)} - \frac{3(x+5)}{(x+5)(x-2)}$$

$$= \frac{2(x-2)-3(x+5)}{(x+5)(x-2)}$$

$$= \frac{2x-4-3x-15}{(x+5)(x-2)}$$

$$= \frac{-x-11}{(x+5)(x-2)}$$

Ex. Simplify $\frac{f(x+h)-f(x)}{h}$ when $f(x) = \frac{1}{x-2}$.

$\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{(x+h)-2} - \frac{1}{x-2}}{h}$; Notice we have fractions in the numerator of a fraction (this is called a complex fraction).

The easiest way to deal with this is to bring the $\frac{1}{h}$ outside as shown here:

$= \frac{1}{h}\left(\frac{1}{(x+h)-2} - \frac{1}{x-2}\right)$; Now subtract the fractions.

$= \frac{1}{h}\left(\frac{(x-2)}{[(x+h)-2](x-2)} - \frac{[(x+h)-2]}{[(x+h)-2](x-2)}\right)$;

$= \frac{1}{h}\left(\frac{(x-2)-[(x+h)-2]}{[(x+h)-2](x-2)}\right)$; Now distribute the minus sign.

$= \frac{1}{h}\left(\frac{(x-2)-x-h+2}{[(x+h)-2](x-2)}\right) = \frac{1}{h}\left(\frac{-h}{[(x+h)-2](x-2)}\right)$; cancel the h's

$= \frac{-1}{[(x+h)-2](x-2)}$.

5. <u>Exponent Rules</u>

Exponent rules are not only important when dealing with exponential functions (which you will use in your Calculus course), but also when taking derivatives of x raised to a power (e.g., finding the derivative of $f(x) = \frac{1}{\sqrt{x^3}}$).

Exponent Rule	Examples
1. $x^{-n} = \frac{1}{x^n}$	$x^{-5} = \frac{1}{x^5}$; $4^{-3} = \frac{1}{4^3} = \frac{1}{64}$
2. $(x^a)(x^b) = x^{a+b}$	$(x^6)(x^4) = x^{10}$; $(3^8)(3^6) = 3^{14}$
3. $x^{\frac{a}{b}} = \sqrt[b]{x^a} = (\sqrt[b]{x})^a$	$x^{\frac{2}{5}} = \sqrt[5]{x^2} = (\sqrt[5]{x})^2$; $x^{\frac{-1}{3}} = \frac{1}{\sqrt[3]{x}}$ $(16)^{\frac{-3}{4}} = \frac{1}{(\sqrt[4]{16})^3} = \frac{1}{2^3} = \frac{1}{8}$
4. $(x^a)^b = x^{ab}$	$(x^3)^5 = x^{15}$; $(2^3)^2 = 2^6 = 64$
5. $(xy)^a = x^a y^a$	$(5x^2)^3 = (5^3)(x^2)^3 = 125x^6$
6. $(\frac{x}{y})^a = \frac{x^a}{y^a}$	$(\frac{2}{3})^4 = \frac{2^4}{3^4} = \frac{16}{81}$
7. $x^0 = 1$; if $x \neq 0$	$(\frac{3}{8})^0 = 1$; $\pi^0 = 1$.

Common Mistakes (please don't do this):

$5(3)^4 \neq (15)^4$

$3(x+4)^6 \neq (3x+12)^6$

$7(1+x)^{-1} \neq (7+7x)^{-1}$; $\quad (7(1+x)^{-1} = \frac{7}{(1+x)})$

$\sqrt{x^2+9} \neq x+3$

$(x+4)^3 \neq x^3+4^3$

$(x^2-6)^{-3} \neq x^{-6}-6^{-3}$

$\frac{1}{7x^2} \neq 7x^{-2} \qquad (\frac{1}{7x^2} = \frac{1}{7}x^{-2})$

$\frac{x}{5} \neq x^{-5} \qquad (\frac{x}{5} = \frac{1}{5}x)$

Ex. Write with a positive exponent

a. $4x^{-3} = \frac{4}{x^3}$

b. $(4x)^{-3} = \frac{1}{(4x)^3} = \frac{1}{(4^3)(x^3)} = \frac{1}{64x^3}$

c. $\frac{1}{4x^{-3}} = \frac{1}{4}x^3$

d. $(\frac{5x}{2})^{-3} = (\frac{2}{5x})^3 = \frac{2^3}{(5x)^3} = \frac{8}{(5^3)(x^3)} = \frac{8}{125x^3}$

e. $\frac{1}{3}(x+6)^{-\frac{2}{5}} = \frac{1}{3(x+6)^{\frac{2}{5}}}$.

When finding derivatives of functions like $f(x) = \frac{1}{x^5}$, it is much easier if we first write it as $f(x) = x^{-5}$. Thus, in certain cases we need to be able to write expressions with negative exponents rather than positive exponents.

Ex. Write with a negative exponent.

a. $\frac{1}{\sqrt[3]{x}^4} = x^{-\left(\frac{4}{3}\right)}$

b. $\frac{1}{3x^4} = \frac{1}{3}\ x^{-4}$

c. $\frac{3}{(4)\sqrt[5]{x^7}} = \frac{3}{4}x^{-\left(\frac{7}{5}\right)}$

d. $\frac{1}{7(x^5-1)^{10}} = \frac{1}{7}(x^5 - 1)^{-10}$

e. $\frac{5}{\sqrt{x^2+1}} = 5(x^2 + 1)^{-\left(\frac{1}{2}\right)}$

Ex. Simplify

a. $\frac{(x^3)(x^2)^4}{\sqrt{x^5}} = \frac{(x^3)(x^8)}{x^{\frac{5}{2}}} = \frac{x^{11}}{x^{\frac{5}{2}}} = x^{11-\frac{5}{2}} = x^{\frac{17}{2}}$

b. $\left(\frac{x^5x^{-2}}{x^{-6}}\right)^3 = \left(\frac{x^3}{x^{-6}}\right)^3 = (x^{3-(-6)})^3 = (x^9)^3 = x^{27}$

6. Inequalities

When is $x^2 - 2x - 15 > 0$?

To solve inequalities like this:

1. Factor the expression completely

2. Find points where the expression is 0

3. If you are allowed to use a graphing calculator, graph the function. You will be able to see where the function is positive and where it's negative.

4. If you are not allowed to use a graphing calculator, test the sign (+ or -) of the expression at points to the left and right of the points where it's 0.

5. The sign of the expression is the same as the sign of the points you found.

Ex. Solve $x^2 - 2x - 15 > 0$.

1. $x^2 - 2x - 15 = (x-5)(x+3)$

2. so $x^2 - 2x - 15 = 0$ when $x = 5$ or $x = -3$

4. ______________|____________________|________________

-3 $\quad\quad\quad\quad$ 5

at x=-4 $(x$-5)(x+3)= (-4-5)(-4+3)= (neg. number)(neg. number)=positive number

at x=0 (x-5)(x+3)=(0-5)(0+3)= (neg. number)(pos. number)=negative number

at x=6 (x-5)(x+3)=(6-5)(6+3)=(pos. number)(pos. number)=positive number

5. $x^2 - 2x - 15$ ++++++++++++++ - - - - - - - - - - - - - - - - +++++++++++++++++

-3 5

So $x^2 - 2x - 15 > 0$ when $x < -3$ or $x > 5$

$x^2 - 2x - 15 < 0$ when $-3 < x < 5$.

Ex. Solve $x^3 - 4x > 0$.

1. $x^3 - 4x = x(x^2 - 4) = x(x+2)(x-2)$

2. so $x^3 - 4x = 0$ when $x = 0$ or $x = -2$ or $x = 2$

4. __________|_______|_______|__________________________

-2 0 2

At $x = -3$:

$x(x+2)(x-2) = -3(-3+2)(-3-2) = (-)(-)(-) =$ Negative

at $x = -1$:

$x(x+2)(x-2) = -1(-1+2)(-1-2) = (-)(+)(-) =$ Positive

at $x = 1$ $x(x+2)(x-2) = 1(1+2)(1-2) = (+)(+)(-) =$ Negative

at $x = 3$ $x(x+2)(x-2) = 3(3+2)(3-2) = (+)(+)(+) =$ Positive

5. - - - - - - - +++++++ - - - - - ++++++++++++++++++

-2 0 2

So $x^3 - 4x > 0$ when $-2 < x < 0$ or $x > 2$

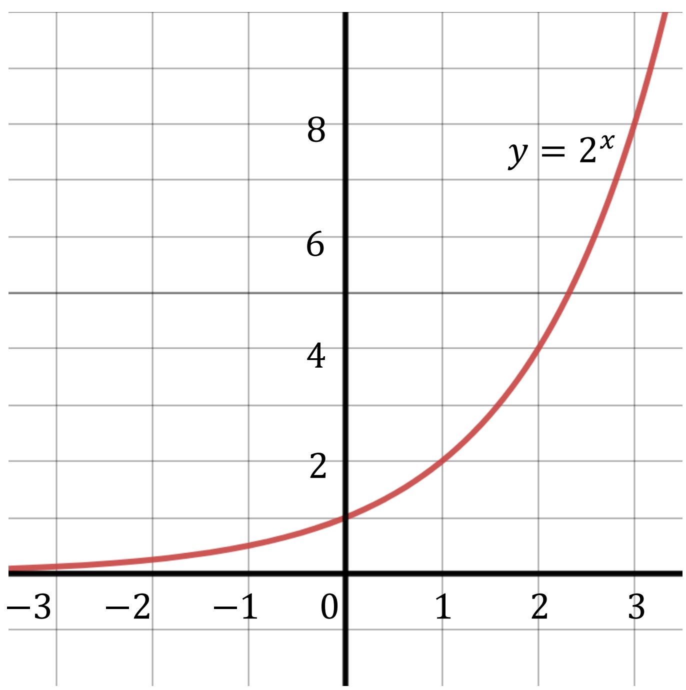
$x^3 - 4x < 0$ when $x < -2$ or $0 < x < 2$.

7. Exponential Functions

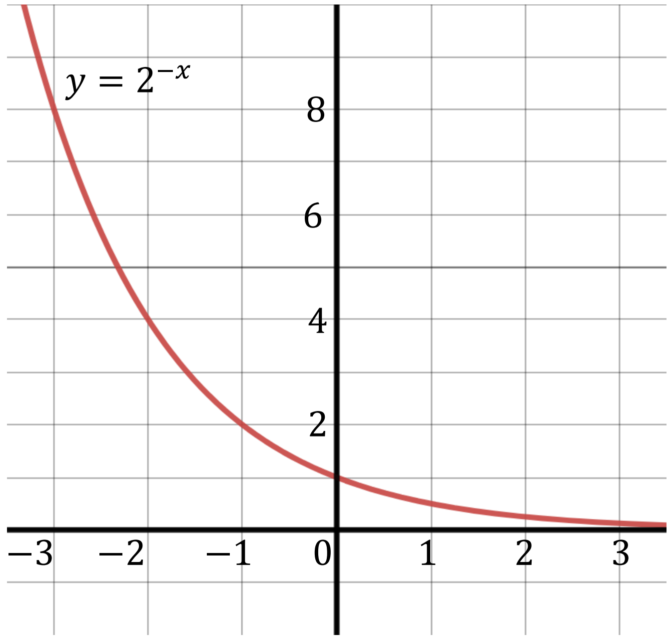
An exponential function is a function where the variable is in the exponent.

Ex. Draw a graph of $y = 2^x$

x	y
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$
-1	$2^{-1} = \frac{1}{2}$
-2	$2^{-2} = \frac{1}{4}$
-3	$2^{-3} = \frac{1}{8}$

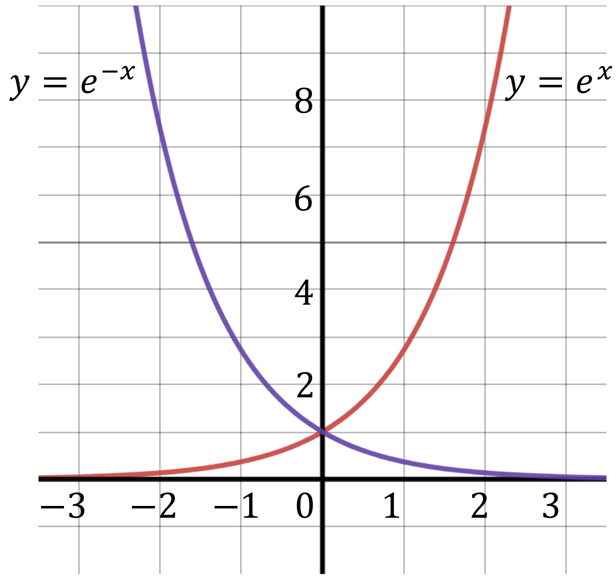


Similarly, we can get a graph of the function $y = 2^{-x}$ which looks like:



In Pre-Calculus you were introduced to the number e≈ 2.718. At the time it might have seemed like a very mysterious number. Now in Calculus you are going to see why it's so important. It turns out that if $f(x) = e^x$, then the derivative of that function is itself, i.e. $f'(x) = e^x$. This is the only function (other than $f(x) = 0$, and $f(x) = Ce^x$) which is equal to its own derivative. This feature means it shows up a lot in mathematics, physics, statistics, when modeling population growth and radioactive decay, and a whole host of other places. Thus when you deal with exponential functions in Calculus, you will generally (but not always) be dealing with e^x as opposed to other bases like 2^x or 10^x.

The graphs of $y = e^x$ and $y = e^{-x}$ are very similar in shape to $y = 2^x$ and $y = 2^{-x}$.



Exponent rules apply regardless of what the base is.

Ex. Simplify:

a. $\sqrt{\frac{e^{5x}}{e^{x}}} = (\frac{e^{5x}}{e^{x}})^{\frac{1}{2}} = (e^{5x-x})^{\frac{1}{2}} = (e^{4x})^{\frac{1}{2}} = e^{2x}$

b. $(\frac{e^{-6x}}{e^{-2x}})^{3} = (e^{(-6x-(-2x))})^{3} = (e^{-4x})^{3} = e^{-12x}$.

8. Logarithms

$y = \log_{10} x$ means $10^y = x$

So, for example, $\log_{10} 100 = 2$ because $10^2 = 100$. Thus it's just the exponent you have to raise 10 to in order to get 100.

Just as you will generally be using exponential functions whose base is e, you will generally be using logarithms to the base e.

$y = \log_e x$ means $e^y = x$.

So, for example, $\log_e 1 = 0$ because $e^0 = 1$ (any number to the 0 power is 1, except 0 to the 0 power).

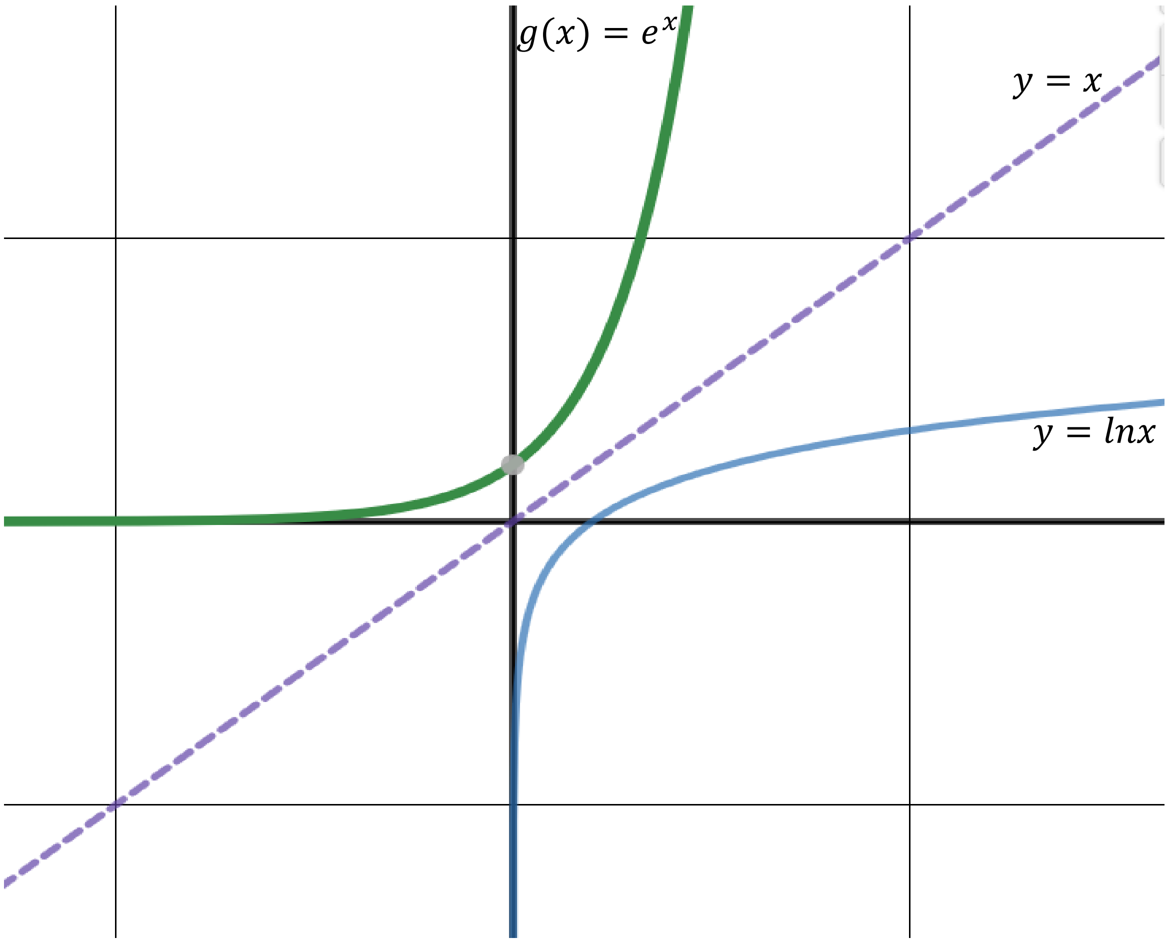
$\log_e x$ is generally written as $\ln x$, which is referred to as the "natural" log of x.

Ex. Simplify

a. $\ln e = 1$ because $e^1 = e$

b. $\ln \sqrt{e} = \ln e^{\frac{1}{2}} = \frac{1}{2}$ because $e^{\frac{1}{2}} = e^{\frac{1}{2}}$.

$f(x) = \ln x$ is the inverse function of $g(x) = e^x$. Thus the graph of $\ln x$ is the reflection over the line $y = x$ of the graph of $g(x) = e^x$.



<u>Function</u>	<u>Domain</u>	<u>Range</u>
$g(x) = e^x$	all real numb.	$y > 0$
$f(x) = \ln x$	$x > 0$	all real numb.

Since $f(x) = \ln x$ and $g(x) = e^x$ are inverse functions of each other it means that $f\big(g(x)\big) = x \;\; and \;\; g\big(f(x)\big) = x$. This gives us the following relationships:

$$f\big(g(x)\big) = f(e^x) = \ln(e^x) = x$$

$$g\big(f(x)\big) = g(\ln x) = e^{\ln x} = x.$$

Ex. Simplify

a. $\ln e^{(3t+5)} = 3t + 5$

b. $e^{3\ln 4} = (e^{\ln 4})^3 = 4^3 = 64.$

Log Rules and Related Exponent Rules:

<u>Log Rules</u>	<u>Related Exponent Rule</u>
$\ln xy = \ln x + \ln y$	$e^{x+y} = (e^x)(e^y)$
$\ln \frac{x}{y} = \ln x - \ln y$	$e^{x-y} = \frac{e^x}{e^y}$
$\ln x^a = a \ln x$	$(e^x)^a = e^{ax}$
$\ln \frac{1}{x} = -\ln x$	$e^{-x} = \frac{1}{e^x}$

Ex. Simplify

a. $\ln 30 + \ln 2 - \ln 20 = \ln 60 - \ln 20 = \ln \frac{60}{20} = \ln 3$

b. $3\ln x$-$\ln 3x + \ln 2 = \ln x^3 - \ln 3x + \ln 2 = \ln \frac{x^3}{3x} + \ln 2 = \ln \frac{2x^2}{3}$

Log rules can also be used to break up a complex function into simpler pieces. This will make it easier to take its derivative.

Ex. Expand

a. $\ln(x^2(x+2)(x-4)^5) = \ln x^2 + \ln(x+2) + \ln(x-4)^5$

$$= 2\ln x + \ln(x+2) + 5\ln(x-4)$$

b. $\ln \frac{e^{3x}\sqrt{(x^2+1)}}{x^4} = \ln \frac{e^{3x}(x^2+1)^{\frac{1}{2}}}{x^4} = \ln e^{3x} + \ln(x^2+1)^{\frac{1}{2}} - \ln x^4$

$$= 3x + \frac{1}{2}\ln(x^2+1) - 4\ln x$$

Some Common Mistakes:

$\ln(x^2+1) \neq \ln x^2 + \ln 1$

$(\ln x)^2 \neq 2\ln x$; $\ln(x^2) = 2\ln x$; $(\ln x)^2 \neq \ln x^2$.

Solving Exponential Equations:

To solve an exponential equation, get the term with the variable in the exponent by itself on one side of the equation and then take the natural log of both sides of the equation.

Ex. Solve $20e^{5t} = 200.$

$$\frac{20e^{5t}}{20} = \frac{200}{20}$$

$$e^{5t} = 10$$

$$\ln e^{5t} = \ln 10$$

$$5t = \ln 10$$

$$t = \frac{\ln 10}{5}.$$