The Weierstrass Theorem

Our next goal is to study $C[a,b]$, the metric space of continuous functions on a closed, bounded interval $[a,b]$ with the metric:

$$d(f,g) = \sup_{a \le x \le b} |f(x) - g(x)|.$$

We want to be able to show that given any continuous function $f \in C[a,b]$ we can find a sequence of polynomials $\{p_n(x)\}$ on $[a,b]$ such that $p_n(x) \to f(x)$ uniformly on $[a,b]$ (which is the same as saying that $p_n(x) \to f(x)$ in the metric space $C[a,b]$). This is useful because it means given any continuous function $f \in C[a,b]$, which could be fairly "complicated", we can approximate it with a polynomial, which is a fairly "simple" function to work with. That is, given any $f \in C[a,b]$ and any $\epsilon > 0$ there is a polynomial, $p(x) \in C[a,b]$ with

$$\sup_{a \le x \le b} |f(x) - p(x)| < \epsilon.$$

This is a common theme in analysis. We often try to approximate complicated functions with less complicated functions. We then prove theorems with the less complicated functions and then show the theorem still holds when you take a limit of less complicated functions that converge to the more complicated function.

Lemma: There is a linear map T from $C[0,1]$ onto $C[a,b]$ such that $d(f,g) = d(T(f),T(g))$ (this is called an **isometry**) and polynomials get mapped to polynomials.

Proof. Let $\sigma\colon [a,b] \to [0,1]$ by $\sigma(x) = \frac{x-a}{b-a}$; for $a \le x \le b$.

σ is a continuous linear function from $[a,b]$ onto $[0,1]$.

We can define: $T\colon C[0,1] \to C[a,b]$

$$T(f) = f \circ \sigma.$$

For example, $\sigma\colon [2,5] \to [0,1]$ by $\sigma(x) = \frac{x-2}{5-2} = \frac{x-2}{3}$; for $2 \le x \le 5$.

If $f(x) = x^2 \in C[0,1]$, then $T(f) = f\big(\sigma(x)\big) = f\left(\frac{x-2}{3}\right) = (\frac{x-2}{3})^2$

Is a continuous function on $[2,5]$.

T is linear because for $f, g \in C[0,1]$ and $a, b \in \mathbb{R}$:

$$T(af+bg) = (af+bg)\big(\sigma(x)\big)$$
$$= af\big(\sigma(x)\big) + bg(\sigma(x))$$
$$= aT(f) + bT(g).$$

Since $\sigma^{-1}(t) = a + t(b-a), \quad 0 \le t \le 1$, we can define an inverse for T by

$$T^{-1}(h) = h(\sigma^{-1}(t))$$

So T is one to one and onto.

Now let's show that T is an isometry (i.e. $d(f,g) = d(T(f),T(g))$).

$$d\big(T(f),T(g)\big) = \sup_{a \le x \le b} |f(\sigma(x)) - g(\sigma(x))| = \sup_{\sigma(a) \le t \le \sigma(b)} |f(t) - g(t)|$$

$$= \sup_{0 \le t \le 1} |f(t) - g(t)| = d(f,g).$$

T also maps polynomials to polynomials:

If $p(t) = \sum_{k=0}^{n} a_k t^k$ then $T(p) = p\big(\sigma(x)\big) = \sum_{k=0}^{n} a_k (\frac{x-a}{b-a})^k$, which is also a polynomial.

This lemma says that for our purposes $C[a,b]$ and $C[0,1]$ are identical. Thus we can focus on $C[0,1]$.

Def. A set A is **separable** if it has a countable dense subset.

Ex. $\mathbb{R}$ is separable because $\mathbb{Q}$ is a countable dense subset.

Ex. $\mathbb{C}$ is separable because $\{a + bi |\ a, b \in \mathbb{Q}\}$ is a countable dense subset.

Theorem: $C[0,1]$ is separable.

Proof. We must show that $C[0,1]$ has a countable dense subset. In other words, we must show that there is a countable subset $A \subseteq C[0,1]$ such that given any $f \in C[0,1]$ and any $\epsilon > 0$ we can find a function $g \in A \subseteq C[0,1]$ such that:

$$\sup_{0 \le x \le 1} |f(x) - g(x)| < \epsilon.$$

Let $\epsilon > 0$ be given and $f \in C[0,1]$.

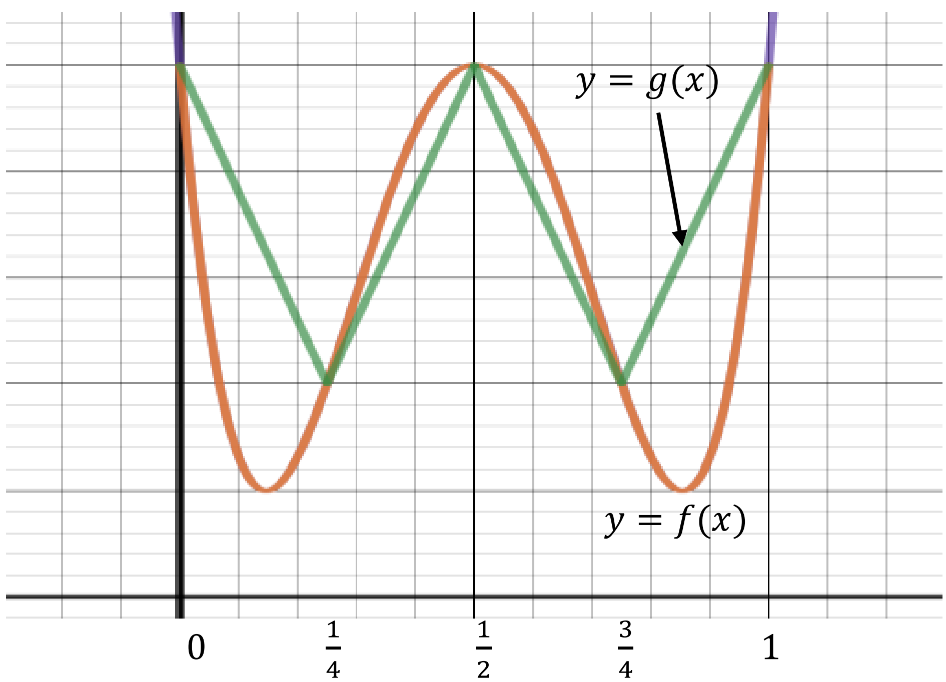
f is continuous on a compact set so it's uniformly continuous. Thus given any $\epsilon > 0$ we can always find an n such that $|f(x) - f(y)| < \frac{\epsilon}{2}$ whenever $|x - y| < \frac{1}{n}$.

Break up $[0,1]$ into n intervals, each of length $\frac{1}{n}$.

Define $g\left(\frac{k}{n}\right) = f\left(\frac{k}{n}\right)$ for $k = 0,1,2, \ldots, n$.

Now let $g(x)$ be linear on $(\frac{k}{n}, \frac{k+1}{n})$.

For example, if $n = 4$ we might have:



Thus $\sup_{0\le x\le 1} |f(x) - g(x)| < \frac{\epsilon}{2}$, since on any subinterval $\frac{k}{n} \le x \le \frac{k+1}{n}$,

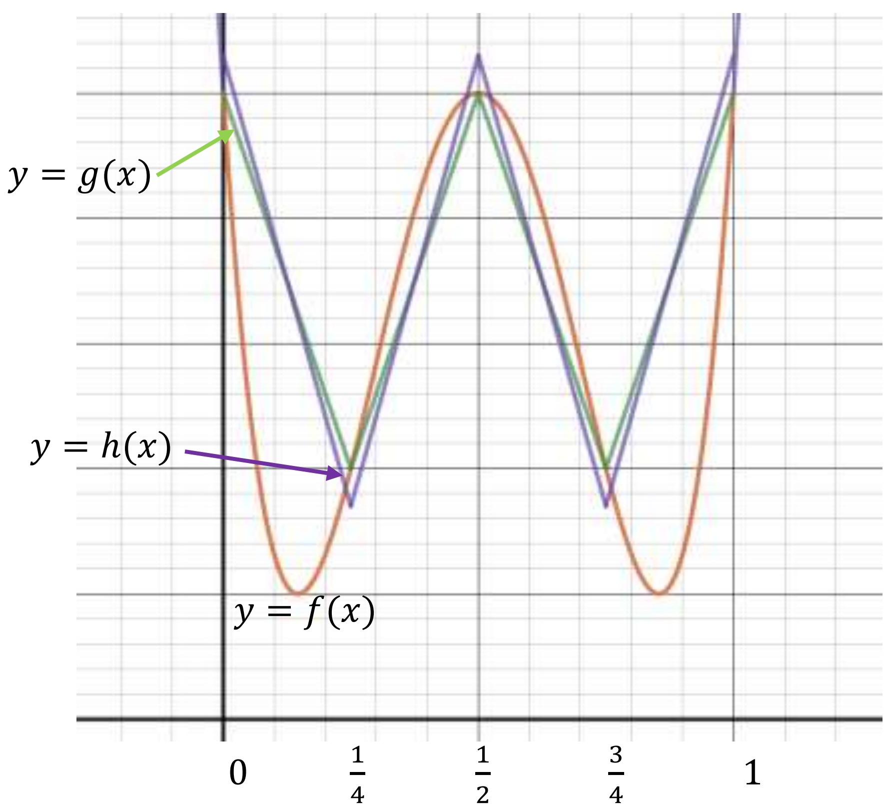
given any point x,

$$\left|f(x) - f\left(\frac{k}{n}\right)\right| < \frac{\epsilon}{2} \quad \text{and} \quad \left|f(x) - f\left(\frac{k+1}{n}\right)\right| < \frac{\epsilon}{2}$$

and $g(x)$ is always between $g\left(\frac{k}{n}\right) = f\left(\frac{k}{n}\right)$ and $g\left(\frac{k+1}{n}\right) = f\left(\frac{k+1}{n}\right)$.

Now let $h(x)$ be defined linearly on the intervals $\frac{k}{n} \le x \le \frac{k+1}{n}$ but at the points $\frac{k}{n}$, $k = 0,1,2,\dots,n$ let $h\left(\frac{k}{n}\right)$ have a rational value such that

$$\left|h\left(\frac{k}{n}\right) - g\left(\frac{k}{n}\right)\right| < \frac{\epsilon}{2} \;; \quad k = 0,1,2,\dots,n.$$



$y = f(x)$ is in orange

$y = g(x)$ is in green

$y = h(x)$ is in purple

Then $\sup_{0\le x\le 1} |h(x) - g(x)| < \frac{\epsilon}{2}$ since $h(x) - g(x)$ is linear on $\frac{k}{n} \le x \le \frac{k+1}{n}$ with

$$\left|h\left(\frac{k}{n}\right) - g\left(\frac{k}{n}\right)\right| < \frac{\epsilon}{2} \quad \text{and} \quad \left|h\left(\frac{k+1}{n}\right) - g\left(\frac{k+1}{n}\right)\right| < \frac{\epsilon}{2} \quad .$$

Now by the triangle inequality:

$$|f(x) - h(x)| \le |f(x) - g(x)| + |g(x) - h(x)| \text{ for all } 0 \le x \le 1.$$

So

$$\sup_{0\le x\le 1} |f(x) - h(x)| \le \sup_{0\le x\le 1} |f(x) - g(x)| + \sup_{0\le x\le 1} |g(x) - h(x)|$$

$$< \quad \frac{\epsilon}{2} \quad + \quad \frac{\epsilon}{2} \quad = \quad \epsilon.$$

The set of all polygonal functions taking only rational values at the nodes $\frac{k}{n}$, $k = 0,1,2, \dots, n$ for some n is countable. This is because for each n, the set of all polygonal functions that have nodes at $\frac{k}{n}$, $k = 0,1,2, \dots, n$ and take on rational values at these points is countable. Then a countable union (over all n) of countable sets is countable.

Weierstrass Approximation Theorem: Given any $f \in C[a, b]$ and $\epsilon > 0$, there is a polynomial p such that $\|f - p\|_\infty = \sup_{a\le x\le b} |f(x) - p(x)| < \epsilon$. Hence there is a sequence of polynomials $\{p_n\}$ such that $p_n \to f$ uniformly on $[a, b]$.

Since $C[a, b]$ and $C[0,1]$ are isometric, and polynomials get mapped to polynomials under our isometry, we only need to prove this theorem for $C[0,1]$. We will actually construct a sequence of polynomials on $[0,1]$, called Bernstein polynomials, that converge uniformly to a given $f \in C[0,1]$ (Bernstein's theorem).

Define $\{B_n(f)\}$; $n = 1,2,\ldots,$ the **Bernstein polynomials** by:

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}; \qquad 0 \le x \le 1,$$

where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.

Ex. Let $f(x) = \sqrt{x}$, find $B_n(f)(x)$ for $n = 2$.

$$B_2(f)(x) = f(0)\left(\frac{2!}{2!\,0!}\right)x^0(1-x)^2 + f\left(\frac{1}{2}\right)\left(\frac{2!}{1!\,1!}\right)x^1(1-x)^1$$

$$+f(1)\left(\frac{2!}{0!2!}\right)x^2(1-x)^0$$

$$= (0)(1)(1-x)^2 + \left(\frac{\sqrt{2}}{2}\right)(2)x(1-x) + (1)(1)x^2$$

$$B_2(f)(x) = \sqrt{2}(x - x^2) + x^2 = \sqrt{2}x + \left(1-\sqrt{2}\right)x^2$$

Let $f_0(x) = 1$, $f_1(x) = x$, $f_2(x) = x^2$.

We will need the following lemma about Bernstein polynomials:

Lemma:

i. $B_n(f_0) = f_0$ and $B_n(f_1) = f_1$.

ii. $B_n(f_2) = \left(1 - \frac{1}{n}\right)f_2 + \frac{1}{n}f_1$. Hence $B_n(f_2) \to f_2$ uniformly on [0,1].

iii. $\sum_{k=0}^{n}\left(\frac{k}{n} - x\right)^2\binom{n}{k}x^k(1-x)^{n-k} = \frac{x(1-x)}{n} \le \frac{1}{4n}$; if $0 \le x \le 1$.

iv. Given $\delta > 0$ and $0 \le x \le 1$, let F denote the set of k in $\{0,1,2,\ldots,n\}$ for which $\left|\frac{k}{n} - x\right| \ge \delta$. Then $\sum_{k\in F}\binom{n}{k}x^k(1-x)^{n-k} \le \frac{1}{4n\delta^2}$.

Proof of i.

i. Recall $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k (b)^{n-k}$

So $(x+(1-x))^n = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1.$

Thus $B_n(f_0) = f_0 = 1.$

To show that $B_n(f_1) = f_1 = x$ notice that

$$\frac{k}{n}\binom{n}{k} = \frac{k}{n}\left(\frac{n!}{(n-k)!k!}\right)$$

$$= \frac{(n-1)!}{(n-k)!(k-1)!}$$

$$= \binom{n-1}{k-1}.$$

This gives us:

$$B_n(f_1) = \sum_{k=0}^{n} \frac{k}{n} \binom{n}{k} x^k (1-x)^{n-k}$$

$$= x \sum_{k=1}^{n} \binom{n-1}{k-1} x^{k-1} (1-x)^{n-k}$$

$$= x \sum_{j=0}^{n-1} \binom{n-1}{j} x^j (1-x)^{(n-1)-j} = x.$$

The Weierstrass approximation theorem follows from the previous lemma by showing that for all $\epsilon > 0$, there exists an N such that if $n \geq N$ then

$$\sup_{0 \leq x \leq 1} |f(x) - B_n f(x)| < \epsilon.$$

Ex. Show that if $f \in C[a,b]$ and $\int_a^b x^n f(x)dx = 0$ for each $n = 0,1,2, \ldots$, then $f(x) = 0$ on $[a,b]$.

The Weierstrass approximation theorem guarantees that there is a sequence of polynomial, $p_n(x)$, that converges uniformly to $f(x)$ on $[a,b]$.

Since $p_n(x) \to f(x)$ uniformly on $[a,b]$, $p_n(x)f(x) \to f^2(x)$ uniformly on $[a,b]$.

Thus we have:

$$\lim_{n\to\infty} \int_a^b p_n(x)f(x)dx = \int_a^b f^2(x)dx.$$

But since $\int_a^b x^m f(x)dx = 0$ for each $m = 0,1,2, \ldots$, then $\int_a^b p_n(x)f(x)dx = 0$ each $n = 0,1,2, \ldots$.

Thus $\int_a^b f^2(x)dx = 0$.

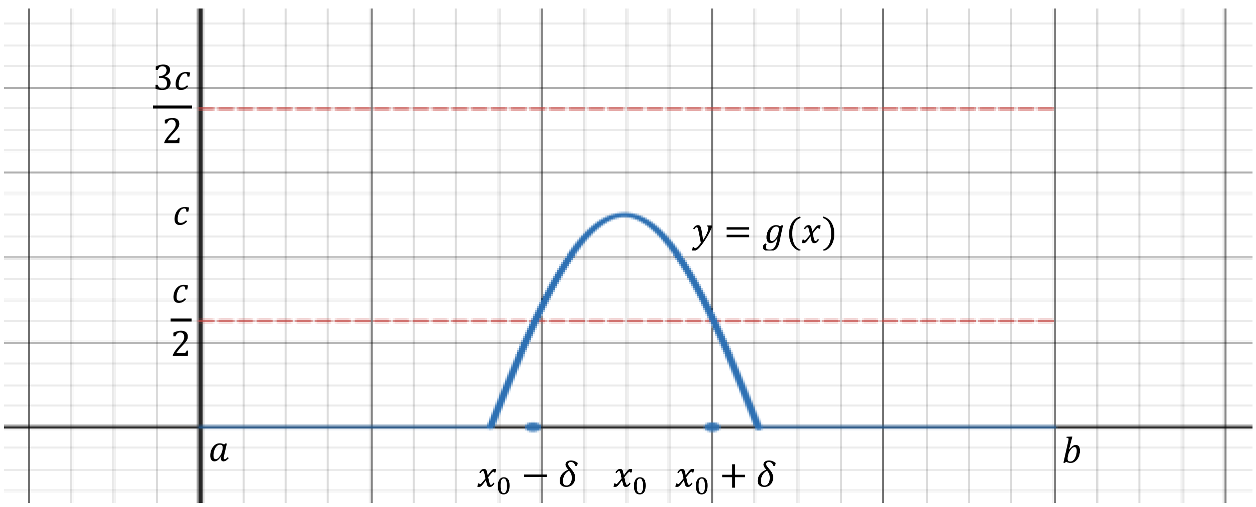
Claim: if $g(x) \geq 0$, and continuous, then $\int_a^b g(x)dx = 0$ implies $g(x) = 0$ on $[a, b]$.

We prove this by contradiction.
Suppose $g(x_0) = c > 0$ for some point $a < x_0 < b$ then by continuity, given $\epsilon = \frac{c}{2}$ there exists a $\delta > 0$ such that for $x_0 - \delta < x < x_0 + \delta$:

$$|g(x) - g(x_0)| < \frac{c}{2}$$

$$-\frac{c}{2} < g(x) - c < \frac{c}{2}$$

$$0 < \frac{c}{2} < g(x) < \frac{3c}{2}.$$



Thus we have: $\int_a^b g(x)dx \geq \int_{x_0-\delta}^{x_0+\delta} g(x)dx > \int_{x_0-\delta}^{x_0+\delta} \frac{c}{2} dx > 0$

which contradicts $\int_a^b g(x)dx = 0$. A similar argument works to show $g(a) = 0$ and $g(b) = 0$.

Thus since $\int_a^b f^2(x)dx = 0$, $f^2(x) \geq 0$ and continuous on $[a, b]$, $f(x) = 0$ on $[a, b]$.

Ex. Show that the Weierstrass approximation theorem is not true for $(0,1)$.

Let $f(x) = \frac{1}{x} \in C(0,1)$.
The Weierstrass approximation theorem says that given any $\epsilon > 0$ there exists a polynomial, $p(x)$, such that $\sup_{0<x<1} |f(x) - p(x)| < \epsilon$.

However, any polynomial $p(x)$ is bounded on $(0,1)$ since it's continuous on $[0,1]$ and $f(x) = \frac{1}{x}$ is unbounded on $(0,1)$ hence $\sup_{0<x<1} |f(x) - p(x)| = \infty$.

Hence there does not exist a polynomial, $p(x)$, such that
$\sup_{0<x<1} |f(x) - p(x)| < \epsilon$
and the Weierstrass approximation theorem is not true for $(0,1)$.

Ex. Show that there is no sequence of polynomials that converge uniformly to $f(x) = e^x$ on $(0, \infty)$.

If there was a sequence of polynomials that converges uniformly to $f(x) = e^x$ on $(0, \infty)$ then given any $\epsilon > 0$ there exists a polynomial, $p(x)$, such that
$\sup_{0<x} |f(x) - p(x)| < \epsilon$.

Claim: For any polynomial $\sup_{0<x} |e^x - p(x)| = \infty$.

$$\sup_{0<x} |e^x - p(x)| = \sup_{0<x} |e^x| \left|1 - \frac{p(x)}{e^x}\right|.$$

Let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x_n$.

By repeated applications of L'Hopital's rule:

$$\lim_{x\to\infty} \frac{p(x)}{e^x} = 0.$$

Thus $\sup_{0<x}|e^x - p(x)| = \sup_{0<x} |e^x| \left|1 - \frac{p(x)}{e^x}\right| = \infty.$

Therefore there does not exist a polynomial on $(0, \infty)$ such that

$$\sup_{0<x}|f(x) - p(x)| < \epsilon.$$