Uniform Convergence of Series: The Weierstrass M-Test

Def. $\sum_{\boldsymbol{i=1}}^{\infty} \boldsymbol{M_i} = \boldsymbol{M}$; where $M_i, M \in \mathbb{R}$, means given:

$S_1 = M_1$

$S_2 = M_1 + M_2$

$S_3 = M_1 + M_2 + M_3$

$\vdots$

$S_n = M_1 + M_2 + M_3 + \cdots + M_n$

then $M = \lim_{n\to\infty} S_n$.

Def. $\boldsymbol{S}(\boldsymbol{x}) = \sum_{\boldsymbol{i=1}}^{\infty} \boldsymbol{f_i}(\boldsymbol{x})$ if given $S_n(x) = \sum_{i=1}^{n} f_i(x)$,

$S_1(x) = f_1(x)$

$S_2(x) = f_1(x) + f_2(x)$

$S_3(x) = f_1(x) + f_2(x) + f_3(x)$

$\vdots$

$S_n(x) = f_1(x) + f_2(x) + f_3(x) + \cdots + f_n(x)$

then $\lim_{n\to\infty} S_n(x) = S(x)$ where this limit means pointwise convergence.

Ex. Let $f_i(x) = \frac{x^{i-1}}{(i-1)!}$; then

$$S_n(x) = \sum_{i=1}^{n} f_i(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!}$$

$$S(x) = \lim_{n\to\infty} S_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots = e^x .$$

Def. We say $\sum_{i=1}^{\infty} \boldsymbol{f_i(x)}$ **converges uniformly to $\boldsymbol{S(x)}$** if the sequence of functions $S_n(x)$ converges uniformly to $S(x)$.

Theorem (Weierstrass M Test): Let $\{f_n(x)\}$ be a sequence of functions on I. Suppose that each $f_n(x)$ is bounded on I, i.e. there exists real numbers M_n such that $|f_n(x)| \leq M_n$ for all $x \in I$. If $\sum_{n=1}^{\infty} M_n$ converges then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on I.

Proof: We know that $S_n(x) = \sum_{i=1}^{n} f_i(x)$ converges uniformly to $S(x)$ if and only if for all $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$, such that for all $x \in I$, if $n, m \geq N$ then $|S_m(x) - S_n(x)| < \epsilon$ (by the theorem we proved in the last section).

Assuming $m > n$:

$$S_m(x) - S_n(x) = \sum_{i=1}^{m} f_i(x) - \sum_{i=1}^{n} f_i(x) = \sum_{i=n+1}^{m} f_i(x).$$

So we need to force $|S_m(x) - S_n(x)| = |\sum_{i=n+1}^{m} f_i(x)| < \epsilon$.

So if we can find a $N \in \mathbb{Z}^+$, such that for all $x \in I$, if $n, m \geq N$ then $|\sum_{i=n+1}^{m} f_i(x)| < \epsilon$ we will have proved $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on I.

Since $\sum_{n=1}^{\infty} M_n$ converges we know given any $\epsilon > 0$ there exists an $N' \in \mathbb{Z}^+$ such that $m, n \geq N'$ implies that

$$|\sum_{i=n+1}^{m} M_i| = M_{n+1} + M_{n+2} + \cdots + M_m < \epsilon.$$

Let $N = N'$. Then we have by the triangle inequality:

$$|S_m(x) - S_n(x)| = |\sum_{i=n+1}^{m} f_i(x)| < |f_{n+1}(x)| + |f_{n+2}(x)| + \cdots |f_m(x)|$$

$$\leq M_{n+1} + M_{n+2} + \cdots + M_m < \epsilon.$$

So $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on I.

Ex. Prove $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges uniformly to $f(x) = e^x$ on the interval $[-k, k]$.

Notice that if we let $f_n(x) = \frac{x^n}{n!}$, then we have:

$|f_n(x)| = |\frac{x^n}{n!}| \le \frac{k^n}{n!} = M_n$ for all $x \in [-k, k]$.

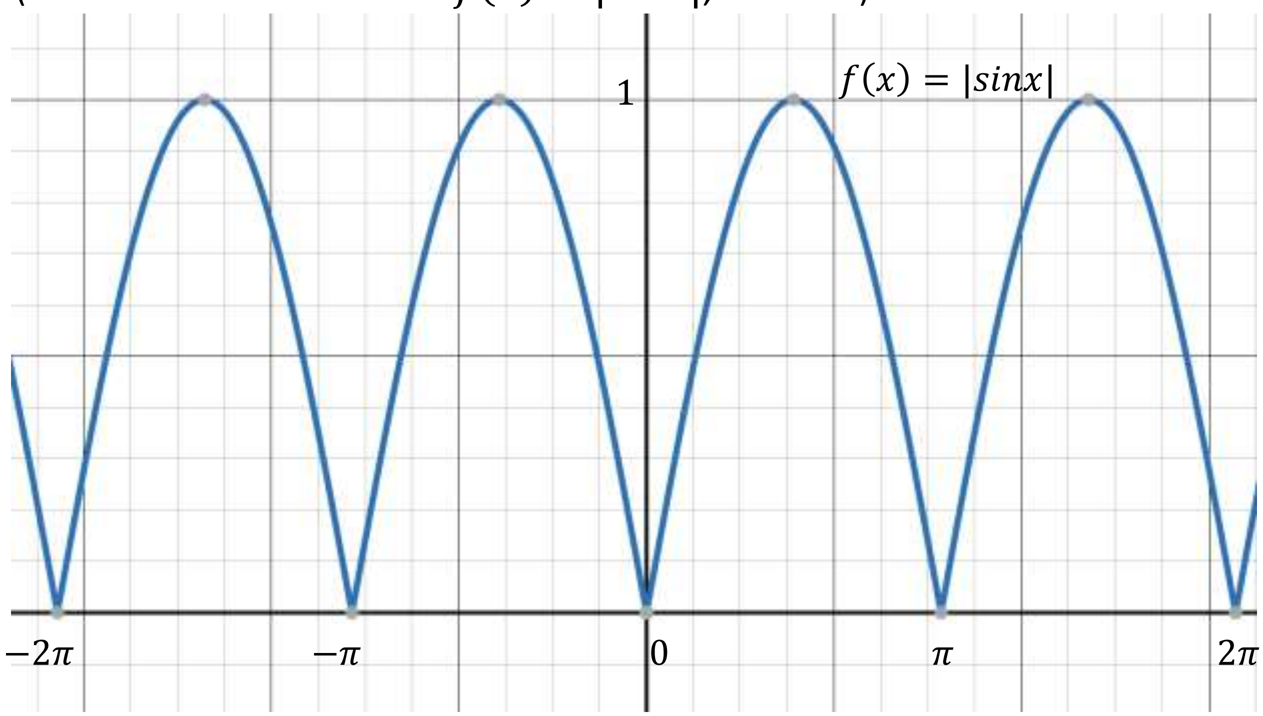
$\sum_{n=0}^{\infty} \frac{k^n}{n!}$ converges by the ratio test since:

$$\lim_{n\to\infty} \left|\frac{M_{n+1}}{M_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(k)^{n+1}}{(n+1)!}}{\frac{k^n}{n!}}\right| = \lim_{n\to\infty} \frac{k}{n+1} = 0 < 1.$$

So by the Weierstrass M Test $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges uniformly on the interval $[-k, k]$. We know from Taylor series that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to $f(x) = e^x$.

Ex. Show that the series $\frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2kx)}{(2k-1)(2k+1)}$ converges uniformly on $\mathbb{R}$.

(This is the Fourier series for $f(x) = |sinx|$; $x \in \mathbb{R}$.)



If we take $f_n(x) = \frac{\cos(2nx)}{(2n-1)(2n+1)}$; then

$|f_n(x)| = |\frac{\cos(2nx)}{(2n-1)(2n+1)}| \leq \frac{1}{4n^2-1} = M_n$ for all $x \in \mathbb{R}$.

$\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$ converges by either the integral test or the comparison test with $\sum_{n=1}^{\infty} \frac{1}{3n^2}$ (which converges because it's a constant multiple of a p-series with $p > 1$).

Thus $\sum_{k=1}^{\infty} \frac{\cos(2kx)}{(2k-1)(2k+1)}$ converges uniformly for all $x \in \mathbb{R}$ by the Weierstrass M-test.

Hence so does $\frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2kx)}{(2k-1)(2k+1)}$.

Ex. Determine where $\sum_{n=1}^{\infty} \frac{x}{n} e^{-nx}$ converges pointwise and uniformly.

For pointwise convergence:

Apply the ratio test.

$$\lim_{n\to\infty} \frac{\left|\left(\frac{x}{n+1}\right)e^{-(n+1)x}\right|}{\left|\frac{x}{n}e^{-nx}\right|} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)\left(\frac{e^{nx}}{e^{(n+1)x}}\right) = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)\left(\frac{1}{e^x}\right) < 1$$

Since $\lim_{n\to\infty} \frac{n}{n+1} = 1$, $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)\left(\frac{1}{e^x}\right) < 1$ when $x > 0$

> 1 when $x < 0$.

So the series converges for $x > 0$ and diverges for $x < 0$.

At $x = 0$; $\sum_{n=1}^{\infty} \frac{x}{n} e^{-nx} = \sum_{n=1}^{\infty} 0 = 0$; so the series converges.

For uniform convergence we want to use the Weierstrass M-test.

Let $f_n(x) = \frac{x}{n}e^{-nx} \geq 0$, for $x \geq 0$ (the series can't converges uniformly for values of x where it diverges).

To use the Weierstrass M-test we need to find an upper bound for $|f_n(x)|$ when $x \geq 0$. So let's find the absolute maximum/minimum values for $f_n(x)$.

$$f_n'(x) = \frac{1}{n}[x(-n)e^{-nx} + e^{-nx}] = \frac{1}{n}(1-nx)e^{-nx} = 0$$

implies $x = \frac{1}{n}$.

$f_n{}'(x)$ goes from positive to negative as x goes through $x = \frac{1}{n}$, so $x = \frac{1}{n}$ is a local maximum. Since this is the only critical point, it must be a global maximum (notice that the function is increasing everywhere on $0 \leq x < \frac{1}{n}$ and decreasing everywhere on $\frac{1}{n} < x < \infty$).

Since $f_n(x) \geq 0$, we have $|f_n(x)| = f_n(x)$. Thus a global maximum of $f_n(x)$ is a global maximum of $|f_n(x)|$.

$$f_n\left(\frac{1}{n}\right) = \frac{\frac{1}{n}}{n}e^{-n\left(\frac{1}{n}\right)} = \frac{1}{n^2}e^{-1} \quad \Longrightarrow \quad \left|f_n(x)\right| = f_n(x) \leq \frac{1}{n^2}e^{-1} = M_n$$

in the Weierstrass M-test.

$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}e^{-1} = e^{-1}\sum_{n=1}^{\infty}\frac{1}{n^2}$; which converges because it's a constant time a p-series with $p > 1$.

Thus $\sum_{n=1}^{\infty}\frac{x}{n}e^{-nx}$ converges uniformly for $x \geq 0$.

Ex. Show $\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}$ converges pointwise for all $|x| \leq 1$, but not uniformly.

$$\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n} = x^2[\frac{1}{1+x^2} + \frac{1}{(1+x^2)^2} + \frac{1}{(1+x^2)^3} + \cdots]$$

$$= \frac{x^2}{1+x^2}[1 + \frac{1}{1+x^2} + \frac{1}{(1+x^2)^2} + \cdots]; \quad \text{[geometric series].}$$

If $x \neq 0$, then $\left|\frac{1}{1+x^2}\right| < 1$, so the sum inside the bracket is $\frac{a}{1-r}$, where $a = 1$ and $r = \frac{1}{1+x^2}$.

$$\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n} = \frac{x^2}{1+x^2}\left[\frac{1}{1-\frac{1}{1+x^2}}\right] = x^2\left[\frac{1}{1+x^2-1}\right] = 1.$$

If $x = 0$, all of the terms are 0, so $\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n} = 0$.

So $\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n} = 1 \qquad$ if $x \neq 0,\ |x| \leq 1$

$= 0 \qquad$ if $x = 0$.

Thus $\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}$ converges for all $|x| \leq 1$.

To show that the convergence is not uniform, we show that the partial sums $S_n(x)$ don't converge uniformly to:

$S(x) = 1 \qquad$ if $x \neq 0,\ |x| \leq 1$

$= 0 \qquad$ if $x = 0$

$$S_n(x) = \sum_{j=1}^{n} \frac{x^2}{(1+x^2)^j} = \frac{x^2}{1+x^2}\left[1 + \frac{1}{1+x^2} + \frac{1}{(1+x^2)^2} + \cdots + \frac{1}{(1+x^2)^{n-1}}\right].$$

Now since $\frac{1-r^n}{1-r} = 1 + r + r^2 + \cdots + r^{n-1}$ we get:

$$S_n(x) = \frac{x^2}{1+x^2}\left[\frac{1-\frac{1}{(1+x^2)^n}}{1-\frac{1}{1+x^2}}\right] = \frac{x^2(1-\frac{1}{(1+x^2)^n})}{1+x^2-1} = 1 - \frac{1}{(1+x^2)^n}\,.$$

For $S_n(x) \to S(x)$ uniformly we would need to show that for all $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$ such that if $n \geq N$ then:

$$|S_n(x) - S(x)| < \epsilon \quad \text{for all } x \text{ with } |x| \leq 1.$$

If $x \neq 0$ and $|x| \leq 1$, we know that $S(x) = 1$. So our epsilon statement becomes:

$$\left|1 - \frac{1}{(1+x^2)^n} - 1\right| = \left|-\frac{1}{(1+x^2)^n}\right| < \epsilon \quad \text{for all } x \neq 0 \text{ with } |x| \leq 1.$$

Now let's show we can't find an N that works for all x with $|x| \leq 1,\ x \neq 0$.

Choose $\epsilon = \frac{1}{2}$.

If you fix an N, no matter how large it is, we can always find a point x with

$|x| \leq 1$, $x \neq 0$ such that $\quad \left|-\frac{1}{(1+x^2)^n}\right| \geq \frac{1}{2}$.

For example, let's show we can always find an x with $\frac{1}{(1+x^2)^n} = \frac{1}{2}$.

$$\frac{1}{(1+x^2)^n} = \frac{1}{2}$$

$$(1+x^2)^n = 2$$

$$1 + x^2 = 2^{\frac{1}{n}}$$

$$x^2 = 2^{\frac{1}{n}} - 1$$

$$x = \pm(2^{\frac{1}{n}} - 1)^{\frac{1}{2}} \qquad \text{(notice that } |x| \le 1, x \ne 0\text{)}$$

So no matter how large N is there is always an $x \neq 0, |x| \leq 1$ where

$$|S_n(x) - S(x)| = \frac{1}{2} \geq \epsilon = \frac{1}{2}.$$

Thus $S_n(x)$ does not converge uniformly to $S(x)$ on $|x| \leq 1$.

Note that $\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}$ does converge uniformly to 1 on $0 < a \leq |x| \leq 1$.

We can see this by:

$$|S_n(x) - S(x)| = \left|1 - \frac{1}{(1+x^2)^n} - 1\right| = \left|-\frac{1}{(1+x^2)^n}\right| = \frac{1}{(1+x^2)^n}.$$

But for $0 < a \leq |x| \leq 1$: $\qquad \frac{1}{(1+x^2)^n} \leq \frac{1}{(1+a^2)^n}$.

So if we can force $\frac{1}{(1+a^2)^n} < \epsilon$ then we can force $|S_n(x) - S(x)| = \frac{1}{(1+x^2)^n} < \epsilon$.

$$\frac{1}{(1+a^2)^n} < \epsilon$$

$$(1+a^2)^n > \frac{1}{\epsilon}$$

$$(n)ln(1+a^2) > \ln(\frac{1}{\epsilon})$$

$$n > \frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(1+a^2)}; \quad \text{for } 0 < a \leq |x| \leq 1.$$

Choose $N > \max(0, \frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(1+a^2)})$, (if $\epsilon > 1$, $\ln\left(\frac{1}{\epsilon}\right) < 0$).

Now if we work the inequalities backward from $n > \frac{\ln\left(\frac{1}{\epsilon}\right)}{\ln(1+a^2)}$, we get:

$$|S_n(x) - S(x)| = \left|1 - \frac{1}{(1+x^2)^n} - 1\right| = \left|-\frac{1}{(1+x^2)^n}\right|$$

$$\leq \frac{1}{(1+a^2)^n} \qquad \text{for } 0 < a \leq |x| \leq 1$$

$$< \epsilon.$$

Note: we could also show that $\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}$ converges uniformly to 1 on $0 < a \leq |x| \leq 1$ with the Weierstrass M-test by letting $f_n(x) = \frac{x^2}{(1+x^2)^n}$ and showing for any $a > 0$, there exists an N such that for $n \geq N$, $|f_n(x)|$ takes its maximum value at $x = a$. Thus we have:

$$|f_n(x)| = \left|\frac{x^2}{(1+x^2)^n}\right| \leq \frac{1}{(1+a^2)^n} = M_n.$$

$\sum_{n=1}^{\infty} M_n$ converges because it's a geometric series with $|r| = \frac{1}{1+a^2} < 1$.

Recall that the radius of convergence for a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is given by $r = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$ if the limit exists (you get this from applying the ratio test to the terms of the power series). This means that given any x such that $|x-a| < r$, the power series will converge absolutely for that value of x.

Theorem: Let r be the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. Then $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges uniformly for all x such that $|x-a| \le \rho < r$.

Proof: Let ρ be any number such that $0 < \rho < r$. Then for any x such that $|x-a| \le \rho$ we have:

$$|c_n(x-a)^n| \le |c_n|\rho^n.$$

A power series converges absolutely for any x with $|x-a| = \rho < r$.

Let $M_n = |c_n|\rho^n$ and $f_n(x) = c_n(x-a)^n$.

Then $|f_n(x)| \le M_n$ and $\sum_{n=0}^{\infty} M_n$ converges.

Thus by the Weierstrass M Test $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges uniformly for all x such that $|x-a| \le \rho < r$.

Theorem: Suppose $\{f_n(x)\}$ is a sequence of functions which are integrable on $[a,b]$ and uniformly converge to $f(x)$, an integrable function on $[a,b]$ then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx.$$

Proof: Since $\{f_n(x)\}$ converges uniformly to $f(x)$ we know given any $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$, such that for all $x \in [a,b]$, if $n \geq N$ then

$$|f_n(x) - f(x)| < \frac{\epsilon}{b-a}.$$

So for $n \geq N$:

$$\begin{aligned} \left| \int_a^b f(x)dx - \int_a^b f_n(x)dx \right| &= \left| \int_a^b \big(f(x) - f_n(x)\big)dx \right| \\ &\leq \int_a^b |f(x) - f_n(x)|dx \\ &\leq \int_a^b \frac{\epsilon}{b-a} dx = \frac{\epsilon}{b-a}(b-a) = \epsilon. \end{aligned}$$

So $\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx$.

Since $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges uniformly for all x such that $|x-a| \leq \rho < r$, where r is the radius of convergence of the power series, it follows from the previous theorem that:

$$\int_p^q \sum_{n=0}^{\infty} c_n(x-a)^n\, dx = \sum_{n=0}^{\infty} \int_p^q c_n(x-a)^n dx$$

as long as $|p-a| < r,\ |q-a| < r$.

This allows us to find Taylor series (i.e. a power series) representation of some function within their radius of convergence.

Ex. Find a power series representation of $f(x) = tan^{-1}(x)$ for $|x| < 1$.

$$tan^{-1}x = \int_{t=0}^{t=x} \frac{1}{1+t^2}\,dt$$
$$= \int_{t=0}^{t=x} (1 - t^2 + t^4 - t^6 + \cdots (-1)^n t^{2n} + \cdots)dt$$

$$= t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \cdots + \frac{(-1)^n t^{2n+1}}{2n+1} + \cdots \Big|_{t=0}^{t=x}$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

Notice that this power series representation does converge at $x = 1$ and gives us an interesting expression for $tan^{-1}1 = \frac{\pi}{4}$.

$$tan^{-1}1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1} + \cdots$$

Theorem: Suppose $\{f_n(x)\}$ is a sequence of functions on $[a, b]$ that converge pointwise to $f(x)$. Suppose that $\{f_n{}'(x)\}$ converges uniformly on $[a, b]$ to a continuous function $g(x)$. Then $f(x)$ is differentiable on $[a, b]$ and $f'(x) = \lim_{n\to\infty} f_n{}'(x)$.

Proof: By the previous theorem:
$$\lim_{n\to\infty} \int_a^x f_n{}'(t)dt = \int_a^x \lim_{n\to\infty} f_n'(t)dt$$
$$\lim_{n\to\infty} \left(f_n(x) - f_n(a)\right) = \int_a^x g(t)dt$$
$$f(x) - f(a) = \int_a^x g(t)dt.$$

Since $g(t)$ is continuous we know from the fundamental theorem of calculus that: $f'(x) = g(x)$.

Thus:

$\{f_n{}'(x)\}$ converges uniformly on $[a, b]$ to a continuous function $g(x) = f'(x)$ and $f'(x) = \lim_{n\to\infty} f_n{}'(x)$.

Theorem: If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ has a radius of convergence of r, then

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

also has a radius of convergence of r.

Proof:

Let $S_n(x) = \sum_{i=0}^{n} c_i (x-a)^i$.

Each $S_n(x)$ is differentiable and $S_n{}'(x) = \sum_{i=1}^{n} i c_i (x-a)^{i-1}$.

If $\{S_n'(x)\}$ converges uniformly in $|x-a| \leq \rho < r$ (we haven't shown this, but it's true), then it converges to a continuous function (since all of the $\{S_n'(x)\}$ are continuous). Thus from our previous theorem:

$$f'(x) = \lim_{n\to\infty} S_n'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}.$$

Ex. This means we can find the Taylor series of $f'(x)$ by differentiating the Taylor series of $f(x)$ term by term and it will have the same radius of convergence as the Taylor series for $f(x)$.

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad for\ |x| < 1$$

$$f'(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}, \qquad for\ |x| < 1$$

$$f''(x) = \frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1) x^{n-1}, \quad for\ |x| < 1.$$