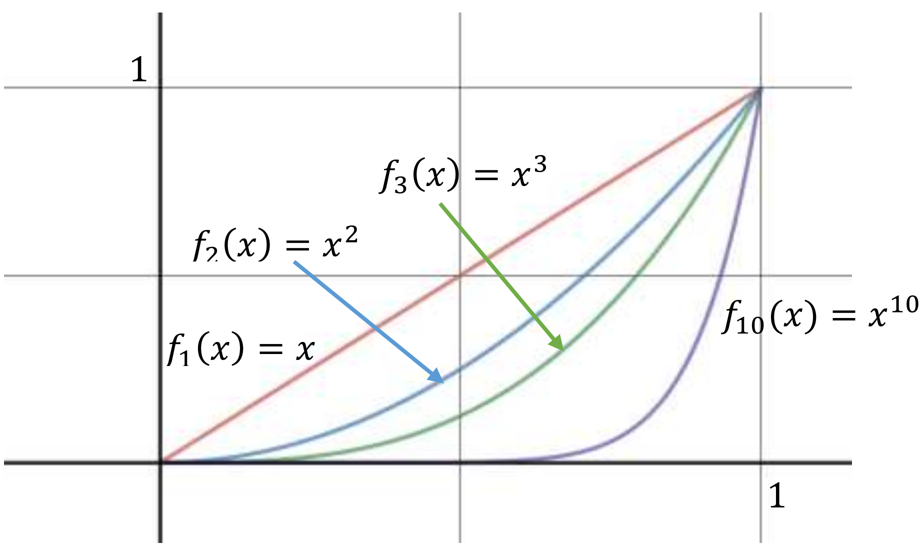
Uniform Convergence

Def. Suppose $\{f_n(x)\}$ is a sequence of functions $f_n: I \subseteq \mathbb{R} \to \mathbb{R}$, where I is an interval (bounded or unbounded, open, closed, or neither) in $\mathbb{R}$. We say **$\{f_n(x)\}$ converges pointwise to $f(x)$,** and write $\lim_{n\to\infty} f_n(x) = f(x)$, if for each $x \in I$, the sequence of real numbers $\{f_n(x)\}$ converges to $f(x)$.

That is, for all $\epsilon > 0$ there exists an $N_x \in \mathbb{Z}^+$ (i.e. N_x can depend on x), such that if $n \geq N_x$ then $|f_n(x) - f(x)| < \epsilon$.

Ex. Let $f_n(x) = x^n$, on $I = [0,1]$. Prove that:

$$\lim_{n\to\infty} f_n(x) = f(x) = 0 \quad if \quad 0 \leq x < 1$$
$$= 1 \quad if \quad x = 1.$$



For example, if $x = \frac{1}{2}$, the sequence $\{f_n(\frac{1}{2})\} = \{\left(\frac{1}{2}\right)^n\} \to 0$ as $n \to \infty$.

However, if $x = 1$, the sequence $\{f_n(1)\} = \{(1)^n\} \to 1$ as $n \to \infty$.

We must show given any $\epsilon > 0$ there exists an $N_x \in \mathbb{Z}^+$, such that if $n \geq N_x$ then $|x^n - f(x)| < \epsilon$.

If $x = 1$, then $|1^n - 1| = 0 < \epsilon$ for any n, so we can choose $N_x = 1$.

If $x = 0$, then $|0^n - 0| = 0 < \epsilon$ for any n, so we again can choose $N_x = 1$.

If $0 < x < 1$, then: $\quad |x^n - 0| < \epsilon$

$$|x|^n < \epsilon$$

$$(n)ln|x| < ln\epsilon$$

$$n > \frac{ln\epsilon}{\ln|x|} \qquad \text{(since } \ln|\mathrm{x}| < 0 \text{ because } 0 < x < 1\text{)}$$

So choose $N_x > \max\left(\frac{ln\epsilon}{ln|x|}, 0\right)$; If $n \geq N_x$ then:

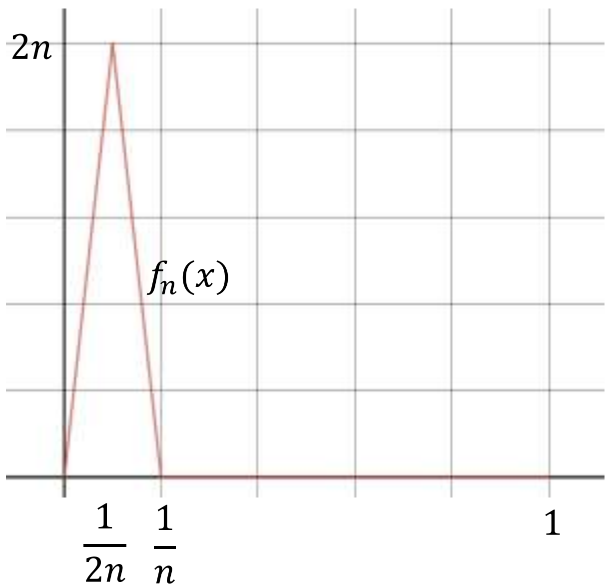
$$|x^n - 0| = |x|^n < |x|^{\frac{ln\epsilon}{\ln|x|}} = (e^{\ln|x|})^{\frac{ln\epsilon}{\ln|x|}} = e^{ln\epsilon} = \epsilon.$$

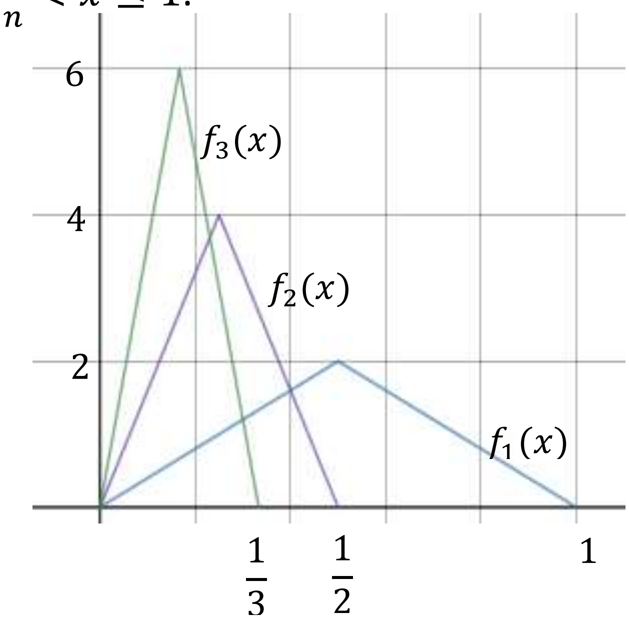
Notice that each $f_n(x)$ in this example is a continuous function, but the sequence of functions converges pointwise to a discontinuous function.

Ex. Let $f_n(x) = 4n^2x$ if $0 \le x \le \frac{1}{2n}$

$= -4n^2x + 4n$ if $\frac{1}{2n} < x \le \frac{1}{n}$

$= 0$ if $\frac{1}{n} < x \le 1.$





Notice that for any $0 \le x \le 1$, $\lim_{n\to\infty} f_n(x) = 0$. That is $f_n(x) \to f(x) = 0$ pointwise on $[0,1]$. Let's prove this.

We must show that for all $\epsilon > 0$, there exists an $N \in \mathbb{Z}^+$ such that if $n \ge N$ then $|f_n(x) - 0| < \epsilon$. Note: the N can depend on the point x.

Notice that if we choose $N > \frac{1}{x}$, when $x \neq 0$, then we have:

$$n \ge N > \frac{1}{x} \quad \text{or equivalently} \quad \frac{1}{n} < x.$$

But if $x > \frac{1}{n}$ then $f_n(x) = 0$ so $|f_n(x) - 0| = |0 - 0| = 0 < \epsilon$.

If $x = 0$, then $f_n(0) = 0$ for all n so for any $N \in \mathbb{Z}^+$, if $n \ge N$ then

$$|f_n(x) - 0| = |0 - 0| = 0 < \epsilon.$$

Thus $f_n(x) \to f(x) = 0$ pointwise on $[0,1]$.

However, notice that:

$\int_0^1 f_n(x)dx = 1$ for all n, so $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 1$.

But $\int_0^1 \lim_{n\to\infty} f_n(x)dx = \int_0^1 f(x)dx = \int_0^1 0dx = 0$.

So we have $\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)dx$.

To try to avoid having a sequence of continuous functions converging to a discontinuous function, or having a sequence of integrable functions whose integrals don't converge to the integral of the limit, we need a "stronger" definition of "convergence".

Def. A sequence of functions $\{f_n(x)\}$, $f_n: I \subseteq \mathbb{R} \to \mathbb{R}$, where I is an interval (bounded or unbounded, open, closed, or neither) in $\mathbb{R}$, **converges uniformly to $\boldsymbol{f(x)}$** if

for all $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$, such that for ALL $x \in I$, if $n \geq N$ then $|f_n(x) - f(x)| < \epsilon$.

1. Notice that for pointwise convergence the N can depend on the point $x \in I$ as well as ϵ. For Uniform convergence the N depends only on ϵ and NOT the point $x \in I$.

2. Uniform convergence is a stronger condition than pointwise convergence. Thus if a sequence of functions converges uniformly to a function $f(x)$, then it must converge pointwise to $f(x)$. However, if a sequence of functions converges pointwise to $f(x)$ then it may, or may not, converge uniformly to $f(x)$.

3. This definition is equivalent to saying that $\lim_{n\to\infty} \sup_{x\in I} |f_n(x) - f(x)| = 0$.
   This gives us another way to prove $\{f_n(x)\}$ does or does not converge uniformly to $f(x)$.

Ex. Show that $\{f_n(x)\}$ converges uniformly to $f(x)$on I if and only if
$\lim_{n\to\infty}\sup_{x\in I}|f_n(x)-f(x)|=0.$

$\Longrightarrow$ Suppose that $\{f_n(x)\}$ converges uniformly to $f(x)$ on I.

We must show that $\lim_{n\to\infty}\sup_{x\in I}|f_n(x)-f(x)|=0.$

That is, given any $\epsilon>0$ there exists an $N\in\mathbb{Z}^+$such that if $n\geq N$ then
$|\sup_{x\in I}|f_n(x)-f(x)|-0|<\epsilon,$ or equivalently, $\sup_{x\in I}|f_n(x)-f(x)|<\epsilon.$

Since $\{f_n(x)\}$ converges uniformly to $f(x)$on I we have:

for all $\epsilon>0$ there exists an $N'\in\mathbb{Z}^+$, such that for all $x\in I$, if $n\geq N'$ then
$|f_n(x)-f(x)|<\frac{\epsilon}{2}$ (since ϵ is an arbitrary positive number we can use $\frac{\epsilon}{2}$).

Choose $N=N'$.

Then $\sup_{x\in I}|f_n(x)-f(x)|\leq\frac{\epsilon}{2}<\epsilon.$

Thus $\lim_{n\to\infty}\sup_{x\in I}|f_n(x)-f(x)|=0.$

$\Longleftarrow$ Suppose $\lim_{n\to\infty}\sup_{x\in I}|f_n(x)-f(x)|=0.$

We must show that $\{f_n(x)\}$converges uniformly to $f(x)$ on I.

That is, for all $\epsilon>0$ there exists an $N\in\mathbb{Z}^+$, such that for all $x\in I$, if $n\geq N$
then $|f_n(x)-f(x)|<\epsilon$.

Since $\lim_{n\to\infty}\sup_{x\in I}|f_n(x)-f(x)|=0$, we know there exists an $N'\in\mathbb{Z}^+$,
such that if $n\geq N'$ then $\sup_{x\in I}|f_n(x)-f(x)|<\epsilon.$

Choose $N = N'$, then we have for all $x \in I$:

$$|f_n(x) - f(x)| \leq \sup_{x \in I} |f_n(x) - f(x)| < \epsilon.$$

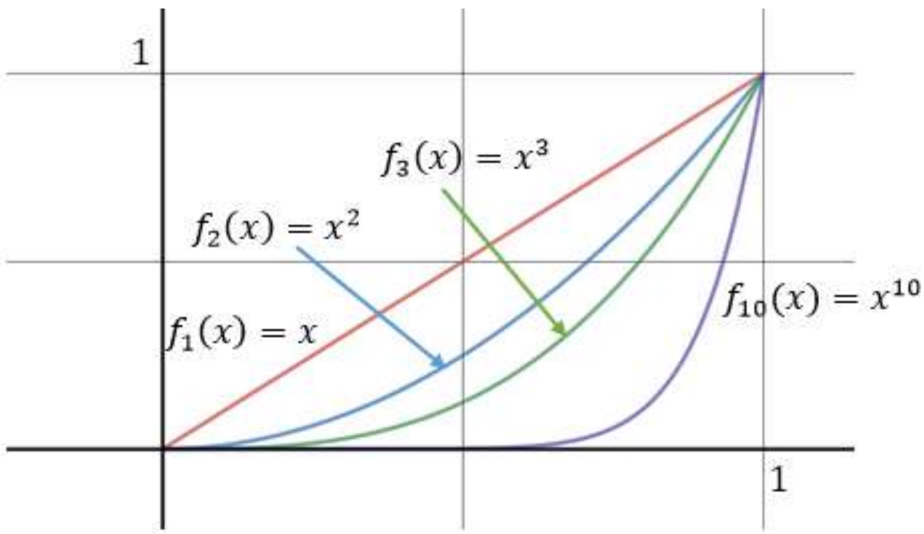
Thus $\{f_n(x)\}$converges uniformly to $f(x)$ on I.

Ex. The sequence of functions $\{x^n\}$ converges pointwise to the function:

$f(x) = 0$ if $0 \leq x < 1$

$= 1$ if $x = 1$

on $I = [0,1]$, but not uniformly.



In the first example we saw that $\{x^n\}$ converges pointwise to $f(x)$. To see that any N we use must depend on the $x \in [0,1]$, notice that if $0 < x < 1$ and we try to solve for an N that will work we get from the epsilon statement:

$$|x^n - 0| < \epsilon \text{ is equivalent to } n > \frac{ln\epsilon}{\ln|x|}$$

Thus if $\epsilon < 1$, as x goes to 1, $\frac{ln\epsilon}{\ln|x|}$ goes to ∞, thus there is no N that will work for all $0 \leq x \leq 1$.

Another way to see this is if we choose $\epsilon = \frac{1}{2}$, given any positive integer n, we can always find an x, where $0 \le x < 1$ and $|x^n - f(x)| = |x^n - 0| \ge \frac{1}{2}$.

$|x^n| \ge \frac{1}{2}$ is equivalent to $x \ge (\frac{1}{2})^{\frac{1}{n}}$ (notice that $0 < (\frac{1}{2})^{\frac{1}{n}} < 1$).

Thus $\lim_{n\to\infty} \sup_{x \in I} |x^n - 0| \ge \frac{1}{2} \implies \lim_{n\to\infty} \sup_{x \in I} |x^n - 0| \ne 0$.

Thus $\{x^n\}$ does not converge uniformly to $f(x) = 0$ on $0 \le x < 1$ or $0 \le x \le 1$.

Notice that if $I = [0, \frac{7}{8}]$ (or $[0,1-\alpha]$, $0 < \alpha \le 1$), $\{x^n\}$ would converge uniformly to $f(x) = 0$. In this case we would just note that: $\left|\frac{ln\epsilon}{\ln|x|}\right| \le \left|\frac{ln\epsilon}{\ln\left|\frac{7}{8}\right|}\right|$

so we could choose $N > \max(\frac{ln\epsilon}{\ln|\frac{7}{8}|}, 0)$ which does not depend on x.

Ex. Show that the sequence of functions $f_n(x) = \frac{\sin(n^2 x)}{n}$ converges uniformly to $f(x) = 0$ for $I = \mathbb{R}$. However, show that $f_n{}'(x)$ does not converge even pointwise to $f'(x)$.

To show that the sequence of functions $f_n(x) = \frac{\sin(n^2 x)}{n}$ converges uniformly to $f(x) = 0$ for $I = \mathbb{R}$, we must show:

for all $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$ (where N doesn't depend on x), such that for all $x \in \mathbb{R}$, if $n \ge N$ then $\left|\frac{\sin(n^2 x)}{n} - 0\right| < \epsilon$.

As usual, we start with the epsilon statement:

$$\left|\frac{\sin(n^2x)}{n} - 0\right| = \left|\frac{\sin(n^2x)}{n}\right| \le \frac{1}{n}$$

So if we can force $\frac{1}{n} < \epsilon$ we're almost done, because $\left|\frac{\sin(n^2x)}{n} - 0\right| \le \frac{1}{n}$.

But $\frac{1}{n} < \epsilon$ is equivalent to $n > \frac{1}{\epsilon}$.

So choose $N > \frac{1}{\epsilon}$ (notice that N depends only on ϵ and not $x \in \mathbb{R}$).

If $\mathrm{n} \ge N > \frac{1}{\epsilon}$ we have:

$$\left|\frac{\sin(n^2x)}{n} - 0\right| = \left|\frac{\sin(n^2x)}{n}\right| \le \frac{1}{n} < \frac{1}{\frac{1}{\epsilon}} = \epsilon.$$

Thus we have shown that $f_n(x) = \frac{\sin(n^2x)}{n}$ converges uniformly to $f(x) = 0$ for $I = \mathbb{R}$.

Now notice that $f_n'(x) = \frac{n^2\cos(n^2x)}{n} = ncos(n^2x)$ and $f'(x) = 0$.

However, for no value of x is $\lim_{n\to\infty} f_n'(x) = 0$, in fact the $\lim_{n\to\infty} f_n'(x)$ does not exist (at least it's not a finite number).

For example, when $x = 0$,

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} n = \infty.$$

Ex. Determine the pointwise limit on the given interval and the intervals on which the convergence is uniform for the following sequences of functions.

a. $f_n(x) = \frac{x}{1+nx^2}$; $\quad x \in \mathbb{R}$.

b. $f_n(x) = \frac{\sqrt{n}x}{1+nx^2}$; $\quad x \in \mathbb{R}$.

a. $f(x) = \lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{x}{1+nx^2} = 0$ for all $x \in \mathbb{R}$ (pointwise convergence)

To test uniform convergence let's find the maximum value of $|f_n(x)|$ on $\mathbb{R}$.

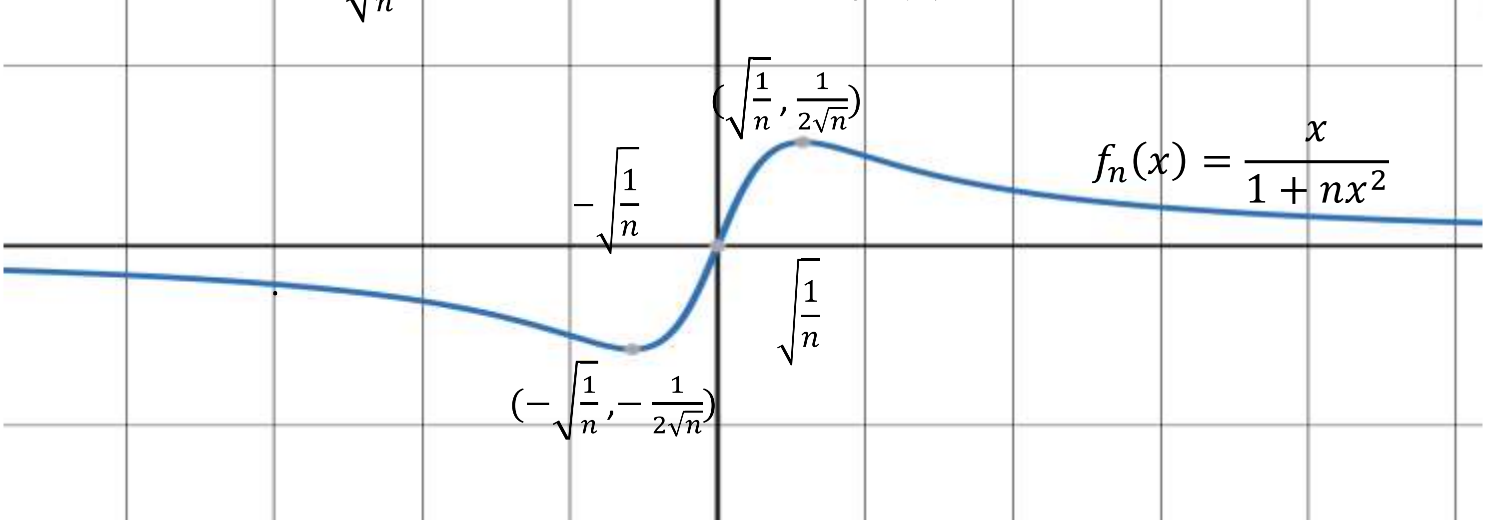
$$f_n'(x) = \frac{(1+nx^2)(1) - x(2nx)}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}.$$

Setting $f_n'(x) = 0$ and solving we get $x = \pm\sqrt{\frac{1}{n}}$.

By checking the sign of $f_n'(x)$ and using $\lim\limits_{x\to\pm\infty} f_n(x) = 0$, we see that:

$x = -\sqrt{\frac{1}{n}}$ yields the minimum value of $f_n(x)$.

$x = \sqrt{\frac{1}{n}}$ yields the maximum value of $f_n(x)$.



$$f_n\left(-\sqrt{\frac{1}{n}}\right) = \frac{-1}{2\sqrt{n}}; \qquad f_n\left(\sqrt{\frac{1}{n}}\right) = \frac{1}{2\sqrt{n}}.$$

So $\sup_{x\in\mathbb{R}}|f_n(x)| = \frac{1}{2\sqrt{n}}$; which goes to 0 as n goes to ∞.

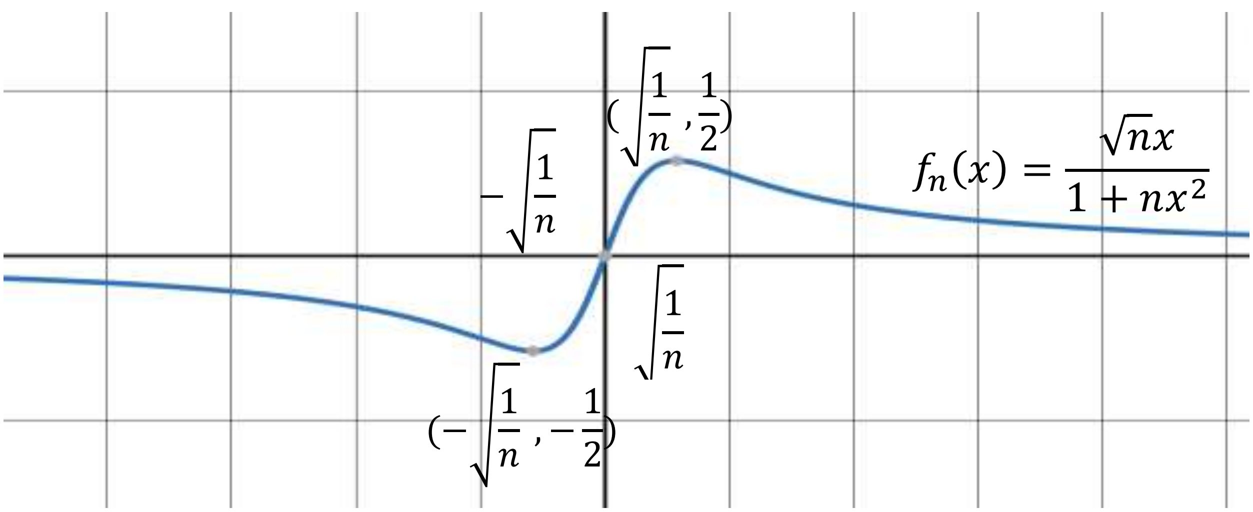
Thus $f_n(x) \to f(x) = 0$ uniformly for all $x \in \mathbb{R}$.

b. $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sqrt{n}x}{1+nx^2}$

$= \lim_{n\to\infty} \frac{\sqrt{n}x}{\sqrt{n}(\frac{1}{\sqrt{n}}+\sqrt{n}x^2)} = 0$ for all $x \in \mathbb{R}$, (pointwise convergence).

$f_n'(x) = \sqrt{n}(\frac{1-nx^2}{(1+nx^2)^2}) = 0,$ when $x = \pm\sqrt{\frac{1}{n}}$.

Once again the max value is at $x = \sqrt{\frac{1}{n}}$ and the min value at $x = -\sqrt{\frac{1}{n}}$.



$$f_n\left(-\sqrt{\frac{1}{n}}\right) = \frac{-1}{2}; \qquad f_n\left(\sqrt{\frac{1}{n}}\right) = \frac{1}{2}.$$

Notice that $\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |f_n(x)| = \frac{1}{2} \neq 0$.

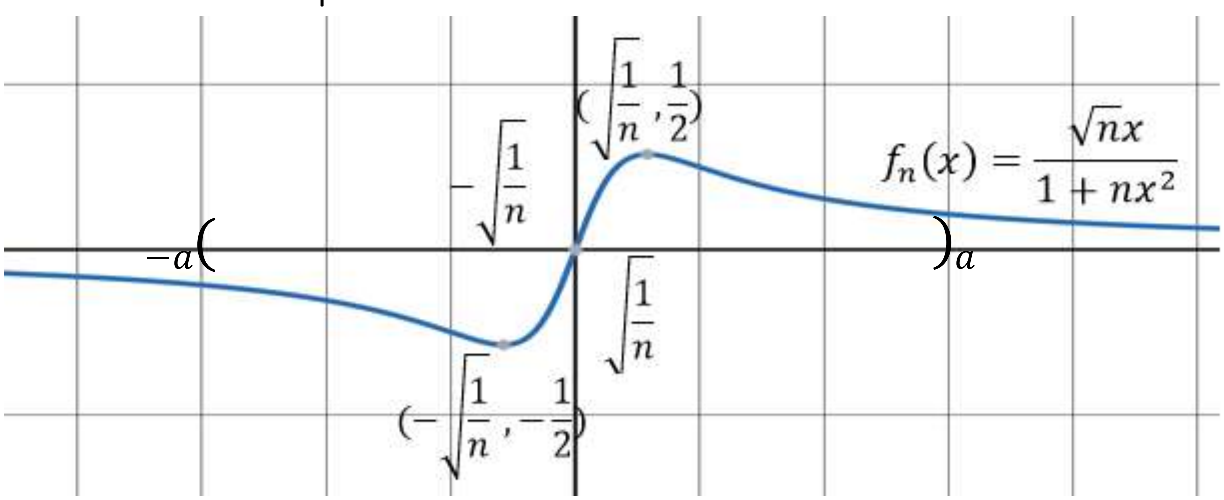
So $f_n(x)$ does not converge uniformly to $f(x) = 0$ for all $x \in \mathbb{R}$.

So where does $f_n(x) \to f(x) = 0$ uniformly?

The problem is that $|f_n(x)|$ takes a maximum value of $\frac{1}{2}$ at

$x = \pm\sqrt{\frac{1}{n}}$ for all n.

However, if we remove any open interval around $x = 0$, eventually $x = \pm\sqrt{\frac{1}{n}}$ will move into that open interval.



Claim: $f_n(x) \to f(x) = 0$ uniformly on any set of the form $\{x \leq -a\} \cup \{x \geq a\},\ a > 0$ (i.e. for $|x| \geq a$).
Note: we could also use $\{x \leq -a\} \cup \{x \geq b\},\ a, b > 0$.

We must show that given any $\epsilon > 0$, there exists an $N \in \mathbb{Z}^+$ such that if $n \geq N$ and $|x| \geq a > 0$ then $\left|\frac{\sqrt{n}x}{1+nx^2} - 0\right| < \epsilon$.

$$\left|\frac{\sqrt{n}x}{1+nx^2} - 0\right| = \left|\frac{\sqrt{n}x}{1+nx^2}\right| = \frac{|x|\sqrt{n}}{|x|(\frac{1}{|x|}+n|x|)} = \frac{\sqrt{n}}{(\frac{1}{|x|}+n|x|)}$$

$$< \frac{\sqrt{n}}{na} = \frac{1}{a\sqrt{n}} < \epsilon$$

$$a\sqrt{n} > \frac{1}{\epsilon}$$

$$\sqrt{n} > \frac{1}{a\epsilon}$$

$$n > \frac{1}{a^2\epsilon^2} .$$

So if we choose $N > \frac{1}{a^2\epsilon^2}$ (notice that N is independent of x) we get:

$$n \geq N > \frac{1}{a^2\epsilon^2}$$

$$\sqrt{n} > \frac{1}{a\epsilon}$$

$$a\sqrt{n} > \frac{1}{\epsilon}$$

$$\frac{\sqrt{n}}{na} = \frac{1}{a\sqrt{n}} < \epsilon.$$

So we have:

$$\left|\frac{\sqrt{n}x}{1+nx^2}-0\right| = \left|\frac{\sqrt{n}x}{1+nx^2}\right| = \frac{|x|\sqrt{n}}{|x|(\frac{1}{|x|}+n|x|)} = \frac{\sqrt{n}}{(\frac{1}{|x|}+n|x|)} < \frac{\sqrt{n}}{na} < \epsilon.$$

Thus $f_n(x) \to f(x) = 0$ uniformly on any set of the form $\{x \leq -a\} \cup \{x \geq a\},\ a > 0$ (i.e. for $|x| \geq a$).

A second way to see that $f_n(x) \to f(x) = 0$ uniformly on any set of the form $S = \{x \leq -a\} \cup \{x \geq a\},\ a > 0$, is to show that $\lim_{n\to\infty} \sup_{x\in S}|f_n(x) - 0| = 0$.

Since $f_n'(x) = \sqrt{n}(\frac{1-nx^2}{(1+nx^2)^2}) = 0$ at $x = \pm\sqrt{\frac{1}{n}}$ we have:

sign of $f_n'(x)$ ——— $-$ ——— | ——— $+$ ——— | ——— $-$ ——— .

$-\sqrt{\frac{1}{n}}$ $\sqrt{\frac{1}{n}}$

Given any positive number $a > 0$, for n sufficiently large, $-a < -\sqrt{\frac{1}{n}} < \sqrt{\frac{1}{n}} < a$.

For these values of n, on the set S, the absolute maximum of $f_n(x)$ occurs at $x = a$ since $f_n'(x) < 0$ and $f_n(x) > 0$ for $x > a$, and the absolute minimum of $f_n(x)$ occurs at $x = -a$ since $f_n'(x) < 0$ and $f_n(x) < 0$ for $x < -a$.

Now notice that:

$$\sup_{x\in S} f_n(x) = f_n(a) = \frac{\sqrt{n}a}{1+na^2}$$

$$\inf_{x\in S} f_n(x) = f_n(-a) = -\frac{\sqrt{n}a}{1+na^2}.$$

Thus we have:

$$\sup_{x\in S} |f_n(x)| = \frac{\sqrt{n}a}{1+na^2}.$$

Now we can say:

$$0 \le \sup_{x\in S} |f_n(x)| = \frac{\sqrt{n}a}{1+na^2} \le \frac{\sqrt{n}a}{na^2} = \frac{1}{\sqrt{n}a}.$$

Thus we have by the squeeze theorem:

$$\lim_{n\to\infty} \sup_{x\in S}|f_n(x) - 0| = 0.$$

Theorem: If $f_n(x)$ converges to $f(x)$ uniformly on an interval $I \subseteq \mathbb{R}$, and $f_n(x)$ is continuous on I for all n, then $f(x)$ is continuous on I.

Proof: we must show that given any point $a \in I$, that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - a| < \delta,\ x \in I,$ then $|f(x) - f(a)| < \epsilon$ (here the δ can depend on the point "a").

Let's start by choosing any point $a \in I$, and fixing any $\epsilon > 0$.

By the triangle inequality we know:

$$|f(x) - f(a)| \le |f(x) - f_n(x)| + |f_n(x) - f(a)|$$

Using the triangle inequality again, but on the 2nd term on the RHS we get:

$$|f_n(x) - f(a)| \le |f_n(x) - f_n(a)| + |f_n(a) - f(a)|$$

Putting these 2 triangle inequalities together we get:

$$|f(x) - f(a)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)|.$$

Now let's show that each one of the terms on the RHS can be made less than $\frac{\epsilon}{3}$.

Since $f_n(x)$ converges to $f(x)$ uniformly we know there exists a $N \in \mathbb{Z}^+$ such that if $n \geq N$ then $|f_n(x) - f(x)| < \frac{\epsilon}{3}$ for any $x \in I$.

Thus the first and the third terms on the RHS can be made less than $\frac{\epsilon}{3}$ by choosing any $n \geq N$, using N in the statement above.

Since $f_n(x)$ is continuous on I we know that given any $\frac{\epsilon}{3} > 0$ there exists a $\delta > 0$ such that if $|x - a| < \delta,\ x \in I,$ then $|f_n(x) - f_n(a)| < \frac{\epsilon}{3}$.

Using this δ we have:

$$
\begin{aligned}
|f(x) - f(a)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}
$$

Thus $f(x)$ is continuous on I.