Uniform Convergence- HW Problems

1. Let $\{f_n\}, \{g_n\}$ be real-valued function on an interval $I \subseteq \mathbb{R}$ such that $\{f_n\}$ converges uniformly to f and $\{g_n\}$ converges uniformly to g on I. Prove that $\{f + g\}$ converges uniformly to $f + g$ on I.

For problems 2-8 find the pointwise limit of the sequence and prove on which interval(s) the sequence converges uniformly.

2. $f_n(x) = \sqrt[n]{x}$ on $[0,1]$
3. $f_n(x) = \frac{1}{1+nx}$ on $[0,\infty)$
4. $f_n(x) = \frac{nx}{1+n^2x^2}$ on $[0,\infty)$
5. $f_n(x) = \frac{\cos(e^n x)}{n}$ on $(-\infty,\infty)$
6. $f_n(x) = \frac{x^2}{x^2+(1-nx)^2}$ on $[0,1]$
7. $f_n(x) = nxe^{-nx}$ on $[0,\infty)$
8. $f_n(x) = \frac{1}{1+x^{2n}}$ on $(-\infty,\infty)$.

9. Let $C[-1,1]$ be the metric space of continuous real-valued functions on $[-1,1]$ with the metric $d(f,g) = \int_{-1}^{1} |f(x) - g(x)|dx$.

Let a sequence of functions in $C[-1,1]$ be given by

$$f_n(x) = \begin{cases} 1 & \text{if } \frac{1}{n} < x \leq 1 \\ \frac{n}{2}x + \frac{1}{2} & \text{if } -\frac{1}{n} \leq x \leq \frac{1}{n} \\ 0 & \text{if } -1 \leq x < -\frac{1}{n} \end{cases}$$

(It may help to sketch the graphs of a few of these functions).

a. Find the pointwise limit of this sequence of functions (ie $\lim_{n\to\infty} f_n(x)$).

b. Show that $\{f_n\}$ is a Cauchy sequence in $C[-1,1]$ with this metric.

c. Does $\{f_n\}$ converge uniformly to its pointwise limit? Explain.

10. Let
$$f_n(x) = \begin{cases} 2nx & \text{if } 0 \leq x \leq \frac{1}{n} \\ \frac{2n(1-x)}{n-1} & \text{if } \frac{1}{n} < x \leq 1. \end{cases}$$

a. Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$.

b. Does f_n converges to f uniformly on $[0,1]$?

11. Let $f_n(x) = \frac{nx}{1+nx}$ on $[0,\infty)$.

a. Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$.

b. Show that $f_n(x)$ does not converge uniformly to $f(x)$ on $(0,\infty)$.