Complete Metric Spaces

Def. A metric space M is said to be **complete** if every Cauchy sequence in M converges to a point in M.

Ex. $\mathbb{R}^n$ is complete with the standard metric

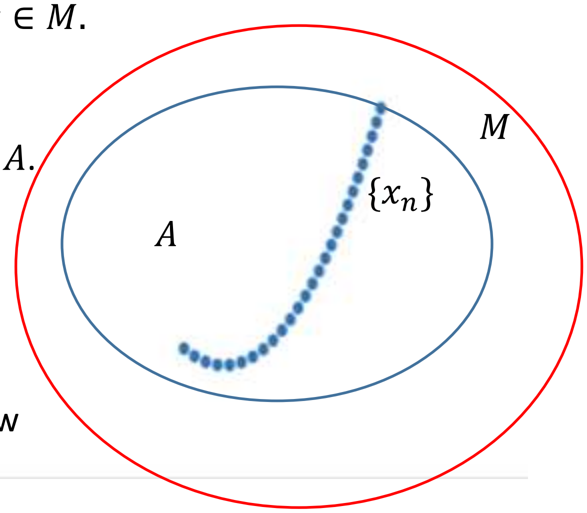
Ex. $(0,1)$ is not complete with the standard metric.

$\mathbb{Q}$ is not complete with the standard metric.

Theorem: Let M, d be a complete metric space and $A \subseteq M$ a subset of M. Then A, d is a complete metric space if and only if A is closed in M.

Proof. Assume that A, d is complete and let $x \in M$ be any limit point of A.

Let $\{x_n\}$ be a sequence in A that converges to $x \in M$.



Since $\{x_n\}$ converges it is a Cauchy sequence in A.

A is complete so $x \in A$.

Hence A is closed.

Now let's assume that A is closed in M and show that A is complete.

Let $\{x_n\}$ be a Cauchy sequence in A.

Then $\{x_n\}$ is also a Cauchy sequence in M, and hence $x_n \to x \in M$, since M is complete.

But A is closed so $x \in A$, and A is complete.

Ex. $[0,1]$, $[0,\infty)$, $\mathbb{Z}$, and $[1,5] \cup \{16\}$ are all complete metric spaces with the standard metric on $\mathbb{R}$.

Notice that if M is complete and totally bounded then every totally bounded sequence in M has a convergent subsequence.

In particular, any closed, bounded subset of $\mathbb{R}$ (i.e. compact subsets of $\mathbb{R}$) is both complete and totally bounded. Thus, for example, every sequence in $[a,b]$ has a convergent subsequence.

How you measure distances can determine whether a metric space is complete. For example, you can have two metric spaces with the same underlying points but one is complete and the other isn't.

Ex. Let $M_1 = [1,\infty)$ with $d_1(x,y) = |x-y|$

$M_2 = [1,\infty)$ with $d_2(x,y) = |\frac{1}{x} - \frac{1}{y}|$.

The sequence $\{1,2,3,4,\ldots\}$ is a Cauchy sequence in M_2 (but not in M_1), but doesn't converge in M_2.

Thus M_1 is a complete metric space (a closed subset of $\mathbb{R}$ with the standard metric), but M_2 is not complete because it has a Cauchy sequence that does not converge in M_2.

Ex. Prove if M is complete then every sequence $\{x_n\}$ in M satisfying $d(x_n, x_{n+1}) < 2^{-n}$, for all n, converges to a point in M.

Since M is complete, we just need to show that $\{x_n\}$ is Cauchy.

So we must show that given any $\epsilon > 0$, there exist an $N \in \mathbb{Z}^+$ such that if

$m, n \geq N$, then $d(x_n, x_m) < \epsilon$.

Assume $m > n$:

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m)$$

$$< 2^{-n} + 2^{-(n+1)} + \cdots + 2^{-(m-1)}$$

$$\leq \sum_{i=n}^{\infty} 2^{-i} = 2^{-n}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) = 2^{-n+1}.$$

So if we can force $2^{-n+1} < \epsilon$ we'll almost be done.

$$2^{-n+1} < \epsilon$$

$$2^{n-1} > \frac{1}{\epsilon}$$

$$(n-1)\ln(2) > \ln\left(\frac{1}{\epsilon}\right) = -\ln(\epsilon)$$

$$n - 1 > \frac{-\ln(\epsilon)}{\ln(2)}$$

$$n > \frac{-\ln(\epsilon)}{\ln(2)} + 1$$

Note: If $\epsilon > 2$ the RHS will be negative, so we

Choose $N > \max(\frac{-\ln(\epsilon)}{\ln(2)} + 1, 0)$.

Now let's show that this N "works".

If $n \geq N > \max(\frac{-\ln(\epsilon)}{\ln(2)} + 1, 0)$

$$n > \frac{-\ln(\epsilon)}{\ln(2)} + 1$$

$$n - 1 > \frac{-\ln(\epsilon)}{\ln(2)}$$

$$(n-1)\ln(2) > -\ln(\epsilon) = \ln\left(\frac{1}{\epsilon}\right)$$

$$\ln\left(2^{(n-1)}\right) > \ln\left(\frac{1}{\epsilon}\right)$$

$$2^{(n-1)} > \frac{1}{\epsilon}$$

$$2^{-n+1} < \epsilon.$$

So if $N > \max(\frac{-\ln(\epsilon)}{\ln(2)} + 1, 0)$, then $d(x_n, x_m) \leq 2^{-n+1} < \epsilon$.

Thus $\{x_n\}$ is a Cauchy sequence.

Ex. It can be shown that if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in a metric space M, d, then $\{d(x_n, y_n)\}$ is a Cauchy sequence in $\mathbb{R}$ (this is a good exercise). Is the converse true? That is, if $\{x_n\}$ and $\{y_n\}$ are sequences in a metric space M, d, and $\{d(x_n, y_n)\}$ is a Cauchy sequence in $\mathbb{R}$ then $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.

This is false!! As a counterexample, Let $\{x_n\}$ and $\{y_n\}$ both be the sequence in $\mathbb{R}, d$ (where d is the standard metric) given by $\{1, 2, 3, \ldots\}$. Then $\{d(x_n, y_n)\}$ is a sequence where all elements are 0, hence clearly a Cauchy sequence, but $\{1, 2, 3, \ldots\}$ is not a Cauchy sequence.

Theorem: For any metric space M, d the following statements are equivalent:

i. M, d is complete.

ii. If $E_1 \supseteq E_2 \supseteq \cdots \supseteq E_n \supseteq \cdots$ is a decreasing sequence of nonempty closed sets in M with $diam(E_n) \to 0$, then $\bigcap_{n=1}^{\infty} E_n \neq \emptyset$ (in fact, it contains exactly one point).

iii. (The Bolzano-Weierstrass Theorem) Every infinite, totally bounded subset of M has a limit point in M.

Ex. We need the $E_n's$ in ii to be closed and $diam(E_n) \to 0$. For example:

Let $E_n = (0, \frac{1}{n})$, then $\bigcap_{n=1}^{\infty} E_n = \emptyset$ (here E_n is not closed)

or let $E_n = [n, \infty)$, then $\bigcap_{n=1}^{\infty} E_n = \emptyset$ (here $diam(E_n) \nrightarrow 0$).

Let V be a vector space.

Def. A **norm** $\|\cdot\|$ on V is a map from $V \to \mathbb{R}$, such that for all $v, w \in V$

a. $\|v\| \geq 0$, and $\|v\| = 0$ if and only if $v = 0$.
b. $\|\lambda v\| = |\lambda| \|v\|$, for any $\lambda \in \mathbb{R}$.
c. $\|v + w\| \leq \|v\| + \|w\|$

We can always define a metric on a vector space V from a norm by:

$$d(v, w) = \|v - w\|.$$

Def. A linear space (i.e. a vector space) that is complete with respect to the distance defined by the norm is called a **Banach space**.

Ex. $\mathbb{R}^n$ is a Banach space with $\|v\| = \sqrt{{x_1}^2 + \cdots + {x_n}^2}$; where $v = < x_1, \dots, x_n >$.