Complete Metric Spaces- HW Problems

1. Let $\{x_n\}, \{y_n\}$ be Cauchy sequences in a metric space X, d. Show that $\{d(x_n, y_n)\}$ is a Cauchy sequence in $\mathbb{R}$ (with the usual metric).

2. Suppose X, d is a complete metric space. Prove that two Cauchy sequences $\{x_n\}, \{y_n\}$ have the same limit if and only if $\lim_{n\to\infty} d(x_n, y_n) = 0$.

3. Find an example of a Cauchy sequence in $(0,1)$ (with the usual metric) and a continuous function $f: (0,1) \to \mathbb{R}$ such that $\{f(x_n)\}$ is not a Cauchy sequence.

4. Show that $\mathbb{R}$ is not complete with the metric

$$d(x, y) = |\tan^{-1} x - \tan^{-1} y|$$

by showing (with an ϵ-N argument) that the sequence $1, 2, 3, \ldots$ is a Cauchy sequence with this metric but the sequence does not converge to any real number (with this metric).

Hint: Use the fact that $\lim_{n\to\infty} \tan^{-1} n = \frac{\pi}{2}$. That is, for all $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$ such that if $n \geq N$ then $|\frac{\pi}{2} - \tan^{-1} x| < \epsilon$ (or $\frac{\epsilon}{2}$). Now write down the definition of $\{n\}$ being a Cauchy sequence with the metric $d(x, y) = |\tan^{-1} x - \tan^{-1} y|$.

5. Find an example of two metrice spaces M_1, d_1 and M_2, d_2 and a continuous function $f: M_1 \to M_2$ such that M_1 is complete but $f(M_1) \subseteq M_2, d_2$ is not complete.

Hint: Think about M_1 and M_2 as having the same underlying points, but d_1 and d_2 are different metrics.

6. Let $C[0,1]$ be the metric space of continuous, real-valued functions on $[0,1]$ with the metric $d(f,g) = \sup_{0 \le x \le 1} |f(x) - g(x)|$.

a. Show that $\{f_n\}$, where $f_n(x) = x^2 + \frac{3}{n}$ is a Cauchy sequence in $C[0,1]$.

b. Let $A = \{f_n\}$. Is A totally bounded? Explain your answer.