Uniform Convergence of Fourier Series

Up to now, we have talked mostly about L_2-convergence of Fourier series. When does $S_n(f)$ converge uniformly to f?

Before we answer this question we need another form of the Cauchy-Schwarz inequality. To do this we first want to consider the set of all real sequences, $x = \{x_n\}$, such that $\sum_{n=1}^{\infty}|x_n|^p < \infty$ for $1 \leq p < \infty$. We call this set $\boldsymbol{\ell_p}$.

In fact, ℓ_p is a vector space under coordinatewise addition and we can define a norm on this vector space by:

$$\|x\|_p = \left(\sum_{n=1}^{\infty}|x_n|^p\right)^{\frac{1}{p}} \quad \text{where } x \text{ is a sequence in } \ell_p$$

To prove uniform convergence of a Fourier series to a function $f(x)$ under the appropriate conditions we will need the following form of the Cauchy-Schwarz inequality for ℓ_2.

Cauchy-Schwarz inequality: $\sum_{i=1}^{\infty}|x_i y_i| \leq \|x\|_2\|y\|_2$

$x, y \in \ell_2$ (Note: $\sum_{i=1}^{\infty} x_i y_i$ is a dot product for ℓ_2)

Proof:

Let's write $< x, y > = \sum_{i=1}^{\infty} x_i y_i$.

Then $< x, x > = \sum_{i=1}^{\infty} x_i^2 = \|x\|_2^2$.

If $t \in \mathbb{R}$, then:

$$0 \leq \|x + ty\|_2^2 = < x + ty, x + ty >$$
$$= < x, x > +2t < x, y > +t^2 < y, y >$$
$$= \|x\|_2^2 + 2t < x, y > +t^2 \|y\|_2^2.$$

This is a quadratic in t that's nonnegative so:

$$At^2 + Bt + C \geq 0$$
$$B^2 - 4AC \leq 0.$$

Or in ths case,

$$(2 < x, y >)^2 - 4\|x\|_2^2 \|y\|_2^2 \leq 0$$
$$< x, y >^2 \leq \|x\|_2^2 \|y\|_2^2$$
$$|< x, y >| \leq \|x\|_2 \|y\|_2$$
$$|\sum_{i=1}^{\infty} x_i y_i| \leq \left(\sum_{i=1}^{\infty} x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} (y_i)^2\right)^{\frac{1}{2}} .$$

The same holds for:

$$x = (|x_1|, |x_2|, |x_3|, \dots)$$
$$y = (|y_1|, |y_2|, |y_3|, \dots), \text{ so}$$
$$\sum_{i=1}^{\infty} |x_i||y_i| \leq \left(\sum_{i=1}^{\infty} x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} y_i^2\right)^{\frac{1}{2}}$$
$$\sum_{i=1}^{\infty} |x_i y_i| \leq \left(\sum_{i=1}^{\infty} x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} y_i^2\right)^{\frac{1}{2}} .$$

Theorem: Let f be a continuous function on $[-\pi, \pi]$ with

$$f(-\pi) = f(\pi)$$

and suppose f has a bounded, piecewise continuous derivative on $[-\pi, \pi]$. Then the Fourier series for f converges uniformly to f on $[-\pi, \pi]$.

Proof: Since $f'(x)$ is piecewise continuous, we can use integration by parts to compare the Fourier coefficients of $f'(x)$ with those of $f(x)$.

$$f'(x) = \frac{{a_0}'}{2} + \sum_{k=1}^{\infty}(a_k' \cos kx + b_k' \sin kx)$$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

For $k \geq 1$ we have:

$$a_k' = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x) \cos kx \ \ dx$$

Let $u = \cos kx$ $\qquad v = f(x)$

$$du = -k \sin kx \ \ dx \qquad dv = f'(x)\, dx$$

$$a_k' = \frac{1}{\pi}[(\cos kx)\, f(x)|_{-\pi}^{\pi} + k\int_{-\pi}^{\pi}(\sin kx)\, f(x) dx]$$

$$= \frac{1}{\pi}[(\cos k\pi)\, f(\pi) - \cos(-k\pi)\, f(-\pi) + k\int_{-\pi}^{\pi}(\sin kx)\, f(x) dx]$$

$f(-\pi) = f(\pi)$ and $\cos(-k\pi) = \cos k\pi$, so

$$a_k' = \frac{k}{\pi}\int_{-\pi}^{\pi} f(x) \sin kx \ \ dx = k b_k.$$

$$a_0' = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x) dx = \frac{1}{\pi}\big(f(\pi) - f(-\pi)\big) = 0.$$

Similarly:

$$b_k' = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin kx \;\, dx$$

Let $u = \sin kx$ $\qquad\qquad v = f(x)$

$du = k\cos kx \;\, dx \qquad dv = f'(x)\,dx$

$$b_k' = \frac{1}{\pi}\left[(\sin kx)\, f(x)|_{-\pi}^{\pi} - k\int_{-\pi}^{\pi} f(x)\cos kx\, dx\right]$$

$$= -\frac{k}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx = -ka_k$$

We know if $g \in R[-\pi,\pi]$, then by Parseval's identity:

$$\frac{c_0^2}{2} + \sum_{k=1}^{\infty}(c_k^2 + d_k^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} \left(g(x)\right)^2 dx < \infty$$

where c_k, d_k are the Fourier coefficients for g. So we can write:

$$\sum_{k=1}^{\infty}(a_k')^2 = \sum_{k=1}^{\infty} k^2 b_k^2 < \infty$$

and

$$\sum_{k=1}^{\infty}(b_k')^2 = \sum_{k=1}^{\infty} k^2 a_k^2 < \infty \;\; .$$

Now we have:

$$\sum_{k=1}^{\infty}|a_k| = \sum_{k=1}^{\infty}\left[(k|a_k|)\cdot\frac{1}{k}\right] \leq \left(\sum_{k=1}^{\infty} k^2 a_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}} < \infty$$

by the Cauchy-Schwarz inequality.

Similarly,

$$\sum_{k=1}^{\infty}|b_k| = \sum_{k=1}^{\infty}\left[(k|b_k|)\cdot\frac{1}{k}\right] \leq (\sum_{k=1}^{\infty}k^2 b_k^2)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}} < \infty\ .$$

We also know:

$$|a_k\cos kx| \leq |a_k|$$

$$|b_k\sin kx| \leq |b_k|.$$

Thus, $|a_k\cos kx + b_k\sin kx| \leq |a_k| + |b_k| = M_k$ for all $x \in [-\pi,\pi]$.

So by the Weierstrass M-test, since:

$$\sum_{k=1}^{\infty}(|a_k| + |b_k|) < \infty$$

$\sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx)$ converges uniformly on $[-\pi,\pi]$.

Hence $\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx)$ converges uniformly on $[-\pi,\pi]$.

Thus, $f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx)$ for $x \in [-\pi,\pi]$

(i.e. the Fourier series converges uniformly on $[-\pi,\pi]$ and hence pointwise on $[-\pi,\pi]$).

Theorem: Termwise Differentiation of Fourier Series

Suppose f and f' are continuous on $[-\pi, \pi]$ with $f(-\pi) = f(\pi)$ and $f'(-\pi) = f'(\pi)$ and suppose that f' has a piecewise continuous derivative on $[-\pi, \pi]$. Then the Fourier series for f' is

$$f'(x) = \sum_{k=1}^{\infty}(-ka_k \sin kx + kb_k \cos kx)$$

This is the termwise differentiation of the Fourier series of f.

Proof: We know from the previous theorem that the Fourier series of $f'(x)$ converges uniformly to $f'(x)$ on $[-\pi, \pi]$. We just need to show that the Fourier coefficients of $f'(x)$ are $-ka_k$ and kb_k.

If $f'(x) = \frac{c_0}{2} + \sum_{k=1}^{\infty}(c_k \cos kx + d_k \sin kx)$

then, $c_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\, dx = \frac{1}{\pi}\big(f(\pi) - f(-\pi)\big) = 0$

$c_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x) \cos kx \; dx.$

As we saw in the previous proof by integrating by parts we get:

$c_k = kb_k$ and $d_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x) \sin kx \, dx = -ka_k$

which is exactly what we get from term by term differentiation of the Fourier series for $f(x)$.

Ex. Solve the differential equation $x''(t) + 4x(t) = F(t)$, where

$$F(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos[(2k-1)t]}{(2k-1)^2}$$

($F(t)$ is the Fourier series for $F(t) = |t|,\ -\pi \le t \le \pi$, and $F(t + 2n\pi) = F(t)$).

To solve this differential equation we first solve the homogeneous equation: $x'' + 4x = 0$.

The general solution to the homogeneous equation is:

$$x_h(t) = A\cos 2t + B sin2t, \quad A, B \in \mathbb{R}.$$

The general solution to $x''(t) + 4x(t) = F(t)$ is given by:

$$x(t) = x_h(t) + x_p(t)$$

where $x_p(t)$ is a particular solution for $x''(t) + 4x(t) = F(t)$.

To find a particular solution for:

$$x''(t) + 4x(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos[(2k-1)t]}{(2k-1)^2}$$

we take:

$$x_p(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt)$$

and take its derivatives and substitute into the differential equation above.

$$x_p{}'(t) = \sum_{k=1}^{\infty}(-ka_k \sin kt + kb_k \cos kt)$$

$$x_p{}''(t) = \sum_{k=1}^{\infty}(-k^2 a_k \cos kt - k^2 b_k \sin kt)$$

$$\sum_{k=1}^{\infty}(-k^2 a_k \cos kt - k^2 b_k \sin kt) +$$

$$4\left(\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt)\right) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos[(2k-1)t]}{(2k-1)^2}$$

$$\frac{4a_0}{2} + \sum_{k=1}^{\infty}(a_k(4-k^2)\cos kt + b_k(4-k^2)\sin kt)$$
$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos[(2k-1)t]}{(2k-1)^2}.$$

So $\frac{4a_0}{2} = \frac{\pi}{2}$ and $a_0 = \frac{\pi}{4}$, $b_k = 0$ for all k and $a_{2k} = 0$ for all k.

$$a_{2k-1}(4-(2k-1)^2) = -\frac{4}{\pi}\left(\frac{1}{(2k-1)^2}\right)$$

$$a_{2k-1} = -\frac{4}{\pi}\left[\frac{1}{(2k-1)^2[4-(2k-1)^2]}\right]$$

$$x_p(t) = \frac{\pi}{8} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos[(2k-1)t]}{(2k-1)^2[4-(2k-1)^2]}$$

So the general solution to $x''(t) + 4x(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos[(2k-1)t]}{(2k-1)^2}$ is:

$$x(t) = x_h(t) + x_p(t)$$
$$= A\cos 2t + B\sin 2t + \frac{\pi}{8} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos[(2k-1)t]}{(2k-1)^2[4-(2k-1)^2]}.$$