Fourier Series: L_2 Convergence and Parseval's Identity

Suppose $f \in R[-\pi,\pi]$, then $f^2 \in R[-\pi,\pi]$ as we saw earlier and:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\big(f(x)\big)^2\ dx \le \frac{1}{\pi}\|f\|_\infty^2\int_{-\pi}^{\pi} 1\, dx = 2\|f\|_\infty^2 < \infty\ .$$

Let's let $T(x) = \frac{c_0}{2} + \sum_{k=1}^{n}(c_k \cos kx + d_k \sin kx)$.

For what values of c_0, c_k and d_k, $k = 1, \dots, n$, is
$\|f - T\|_2^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x) - T(x))^2\, dx$ minimized?

Let's show that for all $n, T(x) = S_n(f)(x)$, where:

$$S_n(f)(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\, dx$$

minimizes $\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x) - T(x))^2\, dx = \|f - T\|_2^2$.

Notice that:

$$\int_{-\pi}^{\pi}(f-T)^2\,dx = \int_{-\pi}^{\pi} f^2\,dx - 2\int_{-\pi}^{\pi} f\;T\,dx + \int_{-\pi}^{\pi} T^2 dx$$

$$\int_{-\pi}^{\pi} f\;T dx = \int_{-\pi}^{\pi} f(x)\left(\frac{c_0}{2} + \sum_{k=1}^{n}(c_k\cos kx + d_k\sin kx)\right) dx$$

$$= \frac{c_0}{2}\int_{-\pi}^{\pi} f(x)\,dx + \sum_{k=1}^{n}\left(c_k\int_{-\pi}^{\pi} f(x)\cos kx\;\,dx\right)$$
$$+\sum_{k=1}^{n}\left(d_k\int_{-\pi}^{\pi} f(x)\sin kx\;\,dx\right)$$

$$= \pi\left[\frac{c_0 a_0}{2} + \sum_{k=1}^{n}(c_k a_k + d_k b_k)\right]$$

$$\int_{-\pi}^{\pi} T^2 dx$$
$$= \int_{-\pi}^{\pi}\left(\frac{c_0}{2} + \sum_{k=1}^{n}(c_k\cos kx + d_k\sin kx)\right)$$
$$\left(\frac{c_0}{2} + \sum_{j=1}^{n}\left(c_j\cos jx + d_j\sin jx\right)\right) dx\,.$$

Since $1, \cos x\,, \sin x\,, \cos 2x, \sin 2x\,, \dots, \cos nx, \sin nx$ are orthogonal and:

$$\int_{-\pi}^{\pi}\cos^2 kx\;dx = \int_{-\pi}^{\pi}\sin^2 kx\;dx = \pi\,, \qquad \int_{-\pi}^{\pi} 1\;dx = 2\pi$$

we get:

$$\int_{-\pi}^{\pi} T^2 dx = \pi\left[\frac{c_0^2}{2} + \sum_{k=1}^{n}(c_k^2 + d_k^2)\right].$$

So we have:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f-T)^2\,dx =$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx - 2\left[\frac{c_0a_0}{2} + \sum_{k=1}^{n}(c_ka_k + d_kb_k)\right] + \left[\frac{c_0^2}{2} + \sum_{k=1}^{n}(c_k^2 + d_k^2)\right]$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f-T)^2 dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\big(f(x)\big)^2\,dx + \frac{c_0^2 - 2a_0c_0}{2}$$
$$+\sum_{k=1}^{n}[(c_k^2 + d_k^2) - 2(c_ka_k + d_kb_k)]\ .$$

Now notice that:

$$c_k^2 - 2c_ka_k = (c_k - a_k)^2 - a_k^2$$
$$d_k^2 - 2d_kb_k = (d_k - b_k)^2 - b_k^2\ .$$

Thus we have:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f-T)^2\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\big(f(x)\big)^2\,dx + \frac{(c_0-a_0)^2}{2} +$$
$$\sum_{k=1}^{n}[(c_k-a_k)^2 + (d_k-b_k)^2] - \frac{a_0^2}{2} - \sum_{k=1}^{n}(a_k^2 + b_k^2).$$

Recall that we want to know how to choose c_0, c_k, d_k in

$T(x) = \frac{c_0}{2} + \sum_{k=1}^{n}(c_k\cos kx + d_k\sin kx)$ to minimize $\frac{1}{\pi}\int_{-\pi}^{\pi}(f-T)^2 dx$.

Clearly, by looking at the RHS of the expression for $\frac{1}{\pi}\int_{-\pi}^{\pi}(f-T)^2\,dx$ we see that $c_k = a_k$ and $d_k = b_k$ minimizes the integral. That is,

$$T(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx) = S_n(f)(x)$$

and the minimum value for the integral is:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f - S_n(f)]^2dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi}(S_n(f)(x))^2\,dx.$$

So if we think of $R[-\pi,\pi]$ with the L_2-norm then of all the possible trigonometric polynomials of degree $\leq n$ in $R[-\pi,\pi]$:

$$\inf_{T\in T_n}\|f-T\|_2 = \|f - S_n(f)\|_2$$

and

$$0 \leq \|f - S_n(f)\|_2^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi}(S_n(f)(x))^2\,dx$$
$$= \|f\|_2^2 - \|S_n(f)\|_2^2.$$

Since $\|f\|_2^2 - \|S_n(f)\|_2^2 \geq 0$ we have:

$$\|S_n(f)\|_2^2 \leq \|f\|_2^2$$

or

$$\|\boldsymbol{S_n(f)}\|_{\boldsymbol{2}} \leq \|\boldsymbol{f}\|_{\boldsymbol{2}}$$ **(Bessel's inequality).**

Now notice that for all n:

$$\|S_n(f)\|_2^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}(S_n(f)(x))^2\,dx = \frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2) \leq \|f\|_2^2 .$$

Thus, if $f \in R[-\pi,\pi]$, its Fourier coefficients are square summable and

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi}\left(f(x)\right)^2\,dx .$$

In particular: $\lim_{k\to\infty} a_k^2 = \lim_{k\to\infty} b_k^2 = 0$; this is a necessary condition for an infinite sum to converge. This then implies that $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 0.$

So the Fourier coefficients of f must tend to 0 as k goes to ∞, i.e.

$$\lim_{k\to\infty} a_k = \lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = 0$$

$$\lim_{k\to\infty} b_k = \lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = 0.$$

This is known as the Riemann lemma (we showed the first part of this earlier by approximating $f(x)$ by step functions).

By Weierstrass' 2nd approximation theorem, we know that if $f \in C^{2\pi}$ then given any $\epsilon > 0$ there is a trig polynomial $T^* \in T_m$, for some $m \in \mathbb{Z}^+$, such that $\|f - T^*\|_\infty < \epsilon$.

Let's use this fact to show that if $f \in C^{2\pi}$ then $S_n(f) \to f$ in the L_2-norm. First notice:

$$\|f\|_2 = \sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx} \leq \sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi}\|f\|_\infty^2\,dx} = (\sqrt{2})\|f\|_\infty.$$

So we have for $n \geq m$:

$$\|f - S_n(f)\|_2 = \inf_{T \in T_n} \|f - T\|_2 \leq \sqrt{2} \inf_{T \in T_n} \|f - T\|_\infty < \epsilon\sqrt{2}$$

Since $T^* \in T_m \subseteq T_n$ and $n \geq m$, $S_n(f) \to f$ in the L_2-norm if $f \in C^{2\pi}$.

Now we want to extend this result to $f \in R[-\pi, \pi]$.

That is, if $f \in R[-\pi, \pi]$, then $\lim_{n \to \infty} \|f - S_n(f)\|_2 = 0$, (i.e. $S_n(f)$ converges to f in the L_2-norm).

First, notice that if $f, g \in R[-\pi, \pi]$ then $S_n(f + g) = S_n(f) + S_n(g)$.

That is, the Fourier coefficients of $f + g$ are the sum of the Fourier coefficients of f and g. For example:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\big(f(x) + g(x)\big)\cos kx\,dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos kx\,dx.$$

In fact, for any $\alpha, \beta \in \mathbb{R}$; $S_n(\alpha f + \beta g) = \alpha S_n(f) + \beta S_n(g)$.

Now suppose $f \in R[-\pi,\pi]$, we know we can find a continuous function $g \in C[-\pi,\pi]$, such that:

$$\int_{-\pi}^{\pi} |f-g| \; dx < \epsilon \qquad \text{(See The Space } R_\alpha[a,b]\text{)}.$$

The same approach will allow us to find a continuous function $g \in C[-\pi,\pi]$ such that:

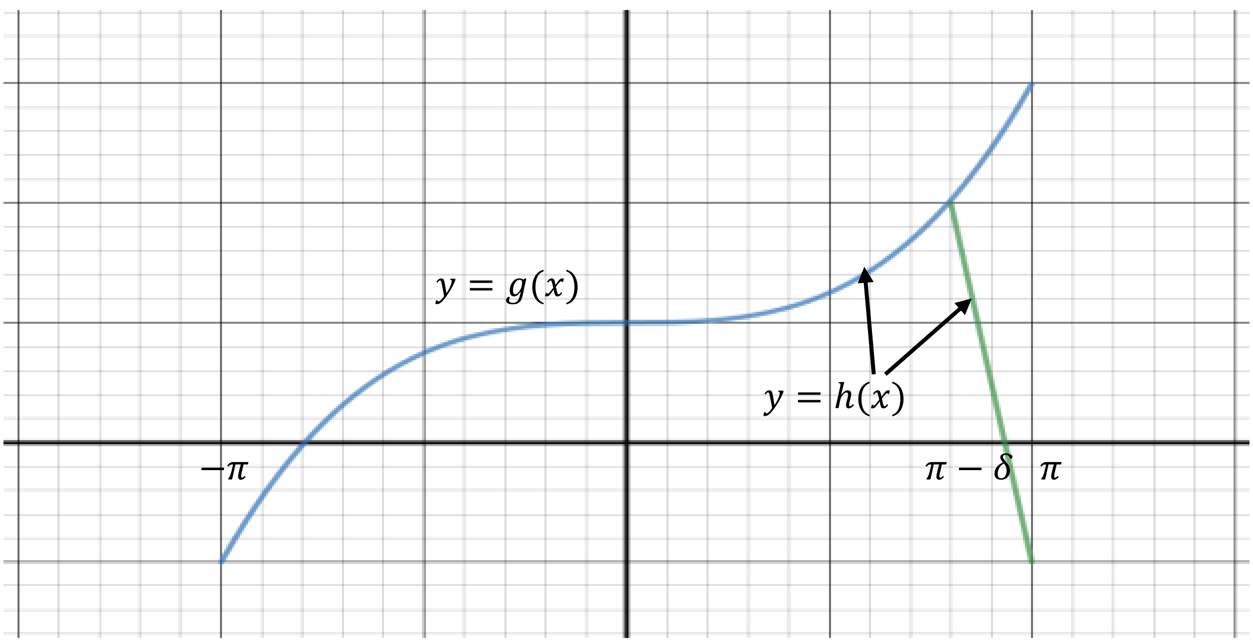
$$\sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} |f-g|^2 \; dx} < \epsilon \,.$$

That is, $\|f-g\|_2 < \epsilon$.

In fact, we can find an $h \in C^{2\pi}$ with:

$$\sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} |f-h|^2 \; dx} < \epsilon$$

or $\|f-h\|_2 < \epsilon$. We can see this since for any $\delta > 0$ we can let $h = g \in C[-\pi,\pi]$ on $[-\pi,\pi-\delta]$, (where $\|f-g\|_2 < \epsilon$), and then let h be linear between $g(\pi-\delta)$ and $h(\pi) = h(-\pi)$.



And if we can do this for any $\epsilon > 0$, we can do it for $\frac{\epsilon}{3}$. So we can find an $h \in C^{2\pi}$ with:

$$\|f - h\|_2 = \left(\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x) - h(x)|^2 \, dx\right)^{\frac{1}{2}} < \frac{\epsilon}{3}.$$

Now by the triangle inequality we have:

$$\|f - S_n(f)\|_2 \leq \|f - h\|_2 + \|h - S_n(f)\|_2$$

Applying the triangle inequality again to the last term on the RHS we get:

$$\begin{aligned}\|h - S_n(f)\|_2 &\leq \|h - S_n(h)\|_2 + \|S_n(h) - S_n(f)\|_2 \\ &= \|h - S_n(h)\|_2 + \|S_n(h - f)\|_2.\end{aligned}$$

And thus,

$$\|f - S_n(f)\|_2 \leq \|f - h\|_2 + \|h - S_n(h)\|_2 + \|S_n(h - f)\|_2$$

From Bessel's inequality:

$$\|S_n(h - f)\|_2 \leq \|h - f\|_2 < \frac{\epsilon}{3}.$$

And since $h \in C^{2\pi}$ for any $\epsilon > 0$, there exists an N such that $n \geq N$ implies

$$\|h - S_n(h)\|_2 < \frac{\epsilon}{3}.$$

So we have: $\|f - S_n(f)\|_2 < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$

Thus, if $f \in R[-\pi, \pi]$ then its Fourier series converges to f in the L_2-norm.

Parseval's Identity:

If $f \in R[-\pi,\pi]$, then $\lim_{n\to\infty} \|f - S_n(f)\|_2 = 0.$

This implies that $\lim_{n\to\infty} \|f - S_n(f)\|_2^2 = 0.$

But we saw earlier that $\|f - S_n(f)\|_2^2 = \|f\|_2^2 - \|S_n(f)\|_2^2.$

So we have: $0 = \lim_{n\to\infty} \|f - S_n(f)\|_2^2 = \lim_{n\to\infty} (\|f\|_2^2 - \|S_n(f)\|_2^2)$.

Thus,
$$\|f\|_2^2 = \lim_{n\to\infty} \|S_n(f)\|_2^2$$

which gives us **Parseval's identity**:

$$\boldsymbol{\frac{1}{\pi}\int_{-\pi}^{\pi}\big(f(x)\big)^2\,dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2).}$$

Parseval's identity tells us that if $f, g \in C^{2\pi}$ have the same Fourier coefficients (this means the Fourier coefficients of $f - g$ are all 0) then f must equal g.

Since if

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\big(f(x) - g(x)\big)\cos kx\,dx = 0$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\big(f(x) - g(x)\big)\sin kx\,dx = 0$$

for all $k \in \mathbb{Z}^+ \cup \{0\}$ then by Parseval's identity:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x) - g(x))^2\,dx = 0.$$

But $(f(x) - g(x))^2$ is a continuous function on $[-\pi,\pi]$ and $(f(x) - g(x))^2 \geq 0.$

Thus $\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x) - g(x))^2\,dx = 0$ implies $f(x) = g(x).$

In contrast, if $f(x) \in C^{\infty}[-\pi, \pi]$ and all of the Taylor series coefficients are 0, we can't conclude $f(x) = 0$ on $[-\pi, \pi]$. For example:

$$f(x) = e^{-\frac{1}{x}} \quad \text{if } x > 0 \text{ and } f(x) = 0 \quad \text{if } x \leq 0.$$

In particular, if a Fourier series for $f \in C^{2\pi}$ converges uniformly, it must converge uniformly to $f(x)$ (and hence pointwise to $f(x)$). This is true because if the Fourier series converges uniformly it must converge to an element $g \in C^{2\pi}$ (since each $S_n(f) \in C^{2\pi}$). But f and g have the same Fourier series and they are both in $C^{2\pi}$ so $f(x) = g(x)$.

Ex. What does Parseval's identity say about the Fourier series for
$f(x) = |x|\ ;\ -\pi \leq x \leq \pi$?

Earlier we found the Fourier series for $f(x) = |x|$ to be:

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos[(2k-1)x]}{(2k-1)^2}\ .$$

We now know that since $f(x) \in C^{2\pi}$, and the Fourier series of f converges uniformly on $[-\pi, \pi]$ (we saw this earlier with the Weierstrass M test), the Fourier series converges pointwise and uniformly on $[-\pi, \pi]$ to $f(x)$.

Parseval's identity says:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left(f(x)\right)^2\,dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \qquad (\,*\,)$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left(f(x)\right)^2\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|^2\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}x^2\,dx = \frac{1}{\pi}\,\frac{x^3}{3}\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{2\pi^3}{3}\right) = \frac{2\pi^2}{3}$$

$$\frac{a_0}{2} = \frac{\pi}{2} \;\Rightarrow\; \frac{a_0^2}{2} = \frac{\pi^2}{2}$$

$$a_{2k-1} = -\frac{4}{\pi}\left(\frac{1}{(2k-1)^2}\right)\;;\quad a_{2k} = 0 \text{ if } k \neq 0$$

Now substituting into ($*$) we get:

$$\frac{2\pi^2}{3} = \frac{\pi^2}{2} + \frac{16}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}$$

$$\frac{\pi^2}{6} = \frac{16}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}$$

$$\frac{\pi^4}{96} = \sum_{k=1}^{\infty}\frac{1}{(2k-1)^4} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots\,.$$

Note: If $f(x)$ has a period of 2π but is defined on an interval other than $[-\pi, \pi]$, say $[c, c+2\pi]$, we have already seen that you can compute the Fourier coefficients by:

$$a_k = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos kx \;\, dx$$

$$b_k = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin kx \;\, dx$$

In this situation, Parseval's identity becomes:

$$\frac{1}{\pi}\int_c^{c+2\pi}\big(f(x)\big)^2\, dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2)\ .$$

In general, if $f(x)$ has a period of $2L$ then Parseval's identity becomes:

$$\frac{1}{L}\int_c^{c+2L}\big(f(x)\big)^2\, dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2)$$

where:

$$a_k = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{k\pi x}{L}\; dx$$

$$b_k = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{k\pi x}{L}\; dx.$$