The Riemann Integral

When does a Riemann -Stieltjes integral reduce to a Riemann integral? In particular, when is

$$\int_a^b f\, d\alpha = \int_a^b f(x)\alpha'(x)\, dx\ ?$$

Theorem: Suppose α is increasing and that α' exists and is a (bounded) Riemann integrable function on $[a, b]$. Then given a bounded function, f on $[a, b]$, we have $f \in R_\alpha[a, b]$ if, and only if, $f\alpha' \in R[a, b]$. In either case,

$$\int_a^b f\, d\alpha = \int_a^b f(x)\alpha'(x)\, dx$$

Proof: Let $\epsilon > 0$ be given and let's show that there exists a partition, P, such that

$$|U_\alpha(f, P) - U(f\alpha', P)| \le \|f\|_\infty \epsilon \qquad (*)$$

and

$$|L_\alpha(f, P) - L(f\alpha', P)| \le \|f\|_\infty \epsilon \qquad (**)$$

By the triangle inequality this will show that $f \in R_\alpha[a, b]$ if, and only if, $f\alpha' \in R[a, b]$ and if either exists then:

$$\int_a^b f\, d\alpha = \int_a^b f(x)\alpha'(x)\, dx.$$

First, let's see why this is true.

Suppose $f \in R_\alpha[a,b]$. Then there exists a partition P such that

$$U_\alpha(f,P) - L_\alpha(f,P) < \epsilon.$$

Then by the triangle inequality we have:

$$|U(f\alpha',P) - L(f\alpha',P)| \le |U(f\alpha',P) - U_\alpha(f,P)|$$
$$+|U_\alpha(f,P) - L_\alpha(f,P)| + |L_\alpha(f,P) - L(f\alpha',P)|.$$

Now using inequalities $(*)$ and $(**)$ we get:

$$|U(f\alpha',P) - L(f\alpha',P)| \le \|f\|_\infty \epsilon + \epsilon + \|f\|_\infty \epsilon$$
$$= (2\|f\|_\infty + 1)\epsilon.$$

Since $2\|f\|_\infty + 1$ is just a constant we have shown that $f\alpha' \in R[a,b]$.

A similar argument will show that if $f\alpha' \in R[a,b]$ then $f \in R_\alpha[a,b]$.

Notice that if both $f\alpha' \in R[a,b]$ and $f \in R_\alpha[a,b]$ then

$$\int_a^b f d\alpha = \inf_P U_\alpha(f,P)$$
$$\int_a^b f\alpha' dx = \inf_P U(f\alpha',P).$$

But $|U_\alpha(f,P) - U(f\alpha',P)| \le \|f\|_\infty \epsilon$

$$\Longrightarrow \int_a^b f d\alpha = \int_a^b f\alpha' dx.$$

Now let’s show:

$$|U_\alpha(f,P) - U(f\alpha',P)| \leq \ \|f\|_\infty \epsilon$$

Since $\alpha' \in R[a,b]$ we know there exists a partition, P, such that:

$$U(\alpha',P) - L(\alpha',P) < \epsilon$$

So we can write: $\sum_{i=1}^{n}(M_i(\alpha') - m_i(\alpha'))\Delta x_i < \epsilon.$

Since α' exists everywhere on $[a,b]$, the mean value theorem guarantees that in each subinterval $[x_{i-1},x_i]$ there is some point $t_i \in [x_{i-1},x_i]$ such that:

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) = (\alpha'(t_i))(x_i - x_{i-1})$$

or

$$\Delta\alpha_i = (\alpha'(t_i))\Delta x_i.$$

Now, if $s_i \in [x_{i-1},x_i]$ is any point then:

$$\sum_{i=1}^{n}|\alpha'(s_i) - \alpha'(t_i)|\Delta x_i \leq \sum_{i=1}^{n}|M_i(\alpha') - m_i(\alpha')|\Delta x_i < \epsilon.$$

Now let $M = \sup\limits_{a\leq x\leq b}|f(x)| = \|f\|_\infty$, then since we know

$$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i = \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\,\Delta\, x_i$$

we get:

$$\begin{aligned}
&\left|\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\,\Delta x_i\right| \\
&\qquad = \left|\sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\,\Delta x_i\right| \\
&\qquad = \left|\sum_{i=1}^{n} f(s_i)(\alpha'(t_i) - \alpha'(s_i))\Delta x_i\right| \leq M\epsilon
\end{aligned}$$

So, we can say:

$$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i \leq \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\,\Delta x_i + M\epsilon \leq \sum_{i=1}^{n} M_i(f\alpha')\,\Delta x_i + M\epsilon$$

or

$$\sum_{i=1}^{n} f(s_i)\,\Delta\alpha_i \leq U(f\alpha', P) + M\epsilon$$

But this is true for any $s_i \in [x_{i-1}, x_i]$, so it's true for $M_i(f)$.

Thus,

$$\sum_{i=1}^{n} M_i(f)\Delta\,\alpha_i \leq U(f\alpha', P) + M\epsilon$$

or

$$U_\alpha(f, P) \leq U(f\alpha', P) + M\epsilon$$

or

$$U_\alpha(f, P) - U(f\alpha', P) \leq M\epsilon.$$

But:

$$\left|\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i\right| \leq M\epsilon$$

also implies,

$$\sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i \leq \sum_{i=1}^{n} f(s_i)\Delta\alpha_i + M\epsilon.$$

So:

$$\sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta\,x_i \leq \sum_{i=1}^{n} f(s_i)\Delta\,\alpha_i + M\epsilon \;\leq \sum_{i=1}^{n} M_i(f)\Delta\,\alpha_i + M\epsilon.$$

$\sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i \leq U_\alpha(f,P) + M\epsilon$ for all $s_i \in [x_{i-1}, x_i]$. Thus it's true for $M_i(f\alpha')$:

$$\sum_{i=1}^{n} M_i(f\alpha')\Delta x_i \leq U_\alpha(f,P) + M\epsilon$$

or

$$U(f\alpha', P) \leq U_\alpha(f,P) + M\epsilon$$

or

$$U(f\alpha', P) - U_\alpha(f,P) \leq M\epsilon$$

thus

$$|U_\alpha(f,P) - U(f\alpha', P)| \leq M\epsilon.$$

A similar argument gives us:

$$|L_\alpha(f,P) - L(\alpha f', P)| \leq M\epsilon.$$

Thus, $f \in R_\alpha[a,b]$ if, and only if, $f\alpha' \in R[a,b]$ and if either exists then:

$$\int_a^b f\, d\alpha = \int_a^b f(x)\alpha'(x)\, dx$$

Ex. Evaluate $\int_0^2 e^{x^2} d\alpha$, where $\alpha(x) = x^2$.

$$\int_a^b f\, d\alpha = \int_a^b f(x)\alpha'(x)\, dx\ ; \qquad f(x) = e^{x^2}, \quad \alpha'(x) = 2x.$$

$$\int_0^2 e^{x^2} d\alpha = \int_0^2 e^{x^2}(2x)dx = e^{x^2}\,|_0^2$$

$$= e^4 - e^0 = e^4 - 1.$$

The probability density function for a normal distribution of mean 0 and standard deviation 1 is given by $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. This means that the probability that a random variable t is less than x is given by:

$$P(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt.$$

If we want to find the expected value of a function, $f(x)$, with respect to a normal distribution of mean 0 and standard deviation 1, we calculate:

$$E[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-\frac{x^2}{2}}\, dx.$$

Since $P(x)$ is differentiable for all x, notice that the expected value of $f(x)$ is just the Riemann-Stieltjes integral of $f(x)$ where $\alpha(x) = P(x)$ and thus
$P'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ (by the fundamental theorem of Calculus).

$$E[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-\frac{x^2}{2}}\, dx = \int_{-\infty}^{\infty} f(x)\, dP.$$

(I'm "cheating" a bit here since the previous theorem is for a closed and bounded interval, but one can get around that problem with appropriate definitions).

Now we want to prove the two fundamental theorems of calculus.

Fundamental Theorem of Calculus I:

Let $f \in R[a, b]$ and for $a \le x \le b$ let $F(x) = \int_{a}^{x} f(t)\, dt$. Then, F is continuous on $[a, b]$. Furthermore, if $f(x)$ is continuous on $[a, b]$, then $F'(x) = f(x)$ for all $x \in [a, b]$.

Proof: Since $f \in R[a,b], f$ is bounded. Suppose $|f(t)| \leq M$ for $a \leq t \leq b$.

If $a \leq x < y \leq b$, then:

$$|F(y) - F(x)| = \left|\int_a^y f(t)\,dt - \int_a^x f(t)\,dt\right|$$

$$= \left|\int_x^y f(t)\,dt\right| \leq M(y-x).$$

So to prove $F(x)$ is continuous at x we need to show given any $\epsilon > 0$ there exists a $\delta > 0$ such that if $|y - x| < \delta$, then $|F(y) - F(x)| < \epsilon$

If we take $\delta = \frac{\epsilon}{M}$ we get:

$$|F(y) - F(x)| \leq M|y - x| < M\delta = M\left(\frac{\epsilon}{M}\right) = \epsilon\ .$$

So, $F(x)$ is continuous at x.

Now assume f is continuous at $x_0 \in [a,b]$. Thus, given any $\epsilon > 0$ there exists a $\delta > 0$ such that if $|t - x_0| < \delta$, then $|f(t) - f(x_0)| < \epsilon$.

Let's show: $F'(x_0) = \lim_{t \to x_0} \frac{F(t)-F(x_0)}{t-x_0} = f(x_0)$.

If $t > x_0$, $|t - x_0| < \delta$ we have,

$$\left|\frac{F(t)-F(x_0)}{t-x_0} - f(x_0)\right| = \left|\frac{1}{t-x_0}\left[\int_a^t f(u)du - \int_a^{x_0} f(u)du\right] - f(x_0)\right|$$

$$= \left|\frac{1}{t-x_0}\int_{x_0}^t f(u)du - \frac{1}{t-x_0}\int_{x_0}^t f(x_0)du\right|$$

$$= \left|\frac{1}{t-x_0}\int_{x_0}^t \big(f(u) - f(x_0)\big)du\right| \leq \frac{1}{t-x_0}\int_{x_0}^t \epsilon\ du$$

$$\left|\frac{F(t)-F(x_0)}{t-x_0} - f(x_0)\right| \leq \frac{1}{t-x_0}(t - x_0)\epsilon\ =\ \epsilon$$

Thus, $F'(x_0) = \lim_{t \to x_0^+} \frac{F(t)-F(x_0)}{t-x_0} = f(x_0)$.

A similar argument shows $F'(x_0) = \lim_{t \to x_0^-} \frac{F(t)-F(x_0)}{t-x_0} = f(x_0)$ for $t < x_0$.

Fundamental Theorem of Calculus II:

If $f \in R[a,b]$ and if there is a differentiable function, F, on $[a,b]$ such that $F' = f$, then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Proof:

Let $\epsilon > 0$ be given. Since $f \in R[a,b]$ there exists a partition,
$P = \{x_0, x_1 \dots, x_n\}$ of $[a,b]$ such that:

$$U(f,P) - L(f,P) < \epsilon$$

By the mean value theorem on $[x_{i-1}, x_i]$, there exists a $t_i \in [x_{i-1}, x_i]$ such that:

$$F(x_i) - F(x_{i-1}) = F'(t_i)(x_i - x_{i-1})$$

or

$$F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i \qquad \text{for } i = 1, \dots, n.$$

Now notice that:

$$\sum_{i=1}^{n} f(t_i)\Delta x_i = \big(F(x_1) - F(x_0)\big) + \big(F(x_2) - F(x_1)\big) + \cdots + (F(x_n) - F(x_{n-1}))$$
$$= F(b) - F(a).$$

but

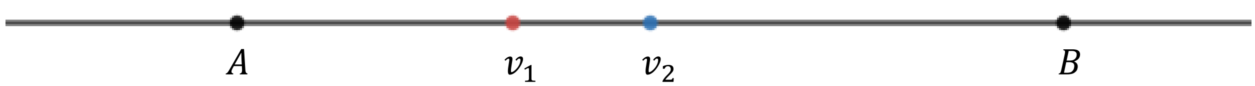
$$L(f,P) \leq \sum_{i=1}^{n} f(t_i)\,\Delta x_i \leq U(f,P)$$

and

$$L(f,P) \leq \int_a^b f(x)dx \leq U(f,P).$$

But notice that for any real numbers $A \leq v_1 \leq B$ and $A \leq v_2 \leq B$ we have:

$$|v_1 - v_2| \leq B - A.$$



Thus we have:

$$\left|\int_a^b f(x)dx - \sum_{i=1}^{n} f(t_i)\Delta x_i\right| \leq U(f,P) - L(f,P).$$

Since $U(f,P) - L(f,P) < \epsilon$, we get:

$$\left|\int_a^b f(x)dx - \sum_{i=1}^{n} f(t_i)\Delta x_i\right| < \epsilon$$

or

$$\left|\int_a^b f(x)dx - \big(F(b) - F(a)\big)\right| < \epsilon.$$

This is true for all $\epsilon > 0$ so,

$$\int_a^b f(x)dx = F(b) - F(a).$$

Ex. Suppose $f(x) \geq 0$ for $x \in [a, b]$ and $f(x)$ is continuous on $[a, b]$ with $\int_a^b f(x)dx = 0$. Prove $f(x) = 0$ on $[a, b]$.

Let $F(x) = \int_a^x f(t)dt$, for all $x \in [a, b]$.

Since $f(t) \geq 0$, $F(x) \geq 0$ for all $x \in [a, b]$.

But $0 = F(b) = \int_a^x f(t)dt + \int_x^b f(t)dt$.

Since $\int_a^x f(t)dt \geq 0$ and $\int_x^b f(t)dt \geq 0$ for all $x \in [a, b]$,

$$F(x) = \int_a^x f(t)dt = 0 \text{ for all } x \in [a, b].$$

By the first fundamental theorem of calculus:

$F'(x) = f(x)$, but $F'(x) = 0$ for all $x \in [a, b]$ because $F(x)$ is constant.

Thus, $f(x) = 0$ for $x \in [a, b]$.

Even though $f(t)$ is not continuous on $[a, b]$ it is still possible $F(x) = \int_a^x f(t)dt$ is differentiable on $[a, b]$.

Ex. On the interval $[-\pi, \pi]$:

Let $f(t) = 2tsin\left(\frac{1}{t}\right) - \cos(\frac{1}{t})$ if $t \neq 0$

$= 0$ if $t = 0.$

Show that $F(x) = \int_0^x f(t)dt$ is differentiable on $[-\pi, \pi]$.

For $x \neq 0$:

$$F'(x) = f(x) = 2xsin\left(\frac{1}{x}\right) - \cos(\frac{1}{x})$$

by the first fundamental theorem of Calculus since $f(t)$ is continuous for $t \neq 0.$

For $x = 0$, since $F(0) = 0$, we have:

$$F'(0) = \lim_{h \to 0} \frac{F(h) - F(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} = 0$$

by the squeeze theorem.

Thus $F(x) = \int_0^x f(t)dt$ is differentiable on $[-\pi, \pi]$ even though $f(t)$ is discontinuous at $t = 0.$

Ex. However, it can happen that if $f(t)$ is discontinuous on

$[a, b]$, then $F(x) = \int_a^x f(t)dt$ is not differentiable on $[a, b]$.

For example, let

$$f(t) = 1 \quad \text{if } 1 \leq t \leq 2$$

$$= 0 \quad \text{if } 0 \leq t < 1.$$

Then we have:

$$F(x) = \int_0^x f(t)dt = x - 1 \quad \text{if } 1 \leq x \leq 2$$

$$= 0 \quad \text{if } 0 \leq x < 1$$

which is not differentiable at $x = 1$.