The Space $R_\alpha[a,b]$

Now we want to understand what kind of structure $R_\alpha[a,b]$ has. Is it a vector space? If so, is there a norm on that vector space and is there a norm that will make it complete?

Theorem: Let $f, g \in R_\alpha[a,b]$ and let $c \in \mathbb{R}$. Then

i. $cf \in R_\alpha[a,b]$ and $\int_a^b cf d\alpha = c\int_a^b f d\alpha$.

ii. $f+g \in R_\alpha[a,b]$ and $\int_a^b (f+g)d\alpha = \int_a^b f d\alpha + \int_a^b g d\alpha$.

iii. $\int_a^b f d\alpha \leq \int_a^b g d\alpha$ whenever $f \leq g$ on $[a,b]$.

iv. $|f| \in R_\alpha[a,b]$ and $|\int_a^b f d\alpha| \leq \int_a^b |f| d\alpha \leq \|f\|_\infty [\alpha(b) - \alpha(a)]$.

v. $fg \in R_\alpha[a,b]$ and $|\int_a^b fg d\alpha| \leq (\int_a^b f^2 d\alpha)^{\frac{1}{2}} (\int_a^b g^2 d\alpha)^{\frac{1}{2}}$.

   This is called the Cauchy-Schwarz inequality.

Proof. If P is any partition of $[a,b]$ and $c \geq 0$ then the supremum of cf on $x_{i-1} \leq x \leq x_i$ is cM_i and the infimum of cf is cm_i.

Thus $U_\alpha(cf,P) = cU_\alpha(f,P)$, $L_\alpha(cf,P) = cL_\alpha(f,P)$ and

$U_\alpha(cf,P) - L_\alpha(cf,P) = c\big(U_\alpha(f,P) - L_\alpha(f,P)\big) = |c|\big(U_\alpha(f,P) - L_\alpha(f,P)\big)$.

If $c < 0$ then $cf = -|c|f$ and:

$$U_\alpha(cf,P) = |c|U_\alpha(-f,P) = -|c|L_\alpha(f,P) = cL_\alpha(f,P)$$

Similarly: $L_\alpha(cf,P) = -|c|U_\alpha(f,P)$

So
$$\begin{aligned} U_\alpha(cf,P) - L_\alpha(cf,P) &= -|c|(L_\alpha(f,P) - U_\alpha(f,P)) \\ &= |c|(U_\alpha(f,P) - L_\alpha(f,P)). \end{aligned}$$

But since $f \in R_\alpha[a,b]$ given any $\epsilon > 0$, there exists a partition, P, such that $\left(U_\alpha(f,P) - L_\alpha(f,P)\right) < \frac{\epsilon}{|c|}$

Thus for that partition:

$$U_\alpha(cf,P) - L_\alpha(cf,P) = |c|\left(U_\alpha(f,P) - L_\alpha(f,P)\right) < |c|\left(\frac{\epsilon}{|c|}\right) = \epsilon \ .$$

So $cf \in R_\alpha[a,b]$.

$$\int_a^b cf d\alpha = \int_a^{\bar{b}} cf\, d\alpha = c\int_a^{\bar{b}} f\, d\alpha \quad \text{if } c \geq 0 \text{ since } U_\alpha(cf,P) = cU_\alpha(f,P)$$

$$= c\int_{\underline{a}}^b f d\alpha \quad \text{if } c < 0 \text{ since } U_\alpha(cf,P) = cL_\alpha(f,P)$$

and $\int_a^{\bar{b}} f\, d\alpha = \int_{\underline{a}}^b f d\alpha = \int_a^b f d\alpha$.

So $\int_a^b cf d\alpha = c\int_a^b f d\alpha$.

ii. Notice that for any partitions P, Q if $f, g \in R_\alpha[a,b]$:

$$L_\alpha(f,P) + L_\alpha(g,Q) \leq L_\alpha(f, P\cup Q) + L_\alpha(g, P\cup Q)$$

$$\leq L_\alpha(f+g, P\cup Q)$$

(since $\inf(f) + \inf(g) \leq \inf(f+g)$)

$$\leq U_\alpha(f+g, P\cup Q)$$

$$\leq U_\alpha(f, P\cup Q) + U_\alpha(g, P\cup Q)$$

(since $\sup(f+g) \leq \sup(f) + \sup(g)$)

$$\leq U_\alpha(f,P) + U_\alpha(g,Q).$$

Since $f, g \in R_\alpha[a,b]$ there exist partitions P, Q such that:

$$U_\alpha(f,P) - L_\alpha(f,P) < \frac{\epsilon}{2}$$

$$U_\alpha(g,Q) - L_\alpha(g,Q) < \frac{\epsilon}{2}.$$

Adding these inequalitites we get:

$$(U_\alpha(f,P) + U_\alpha(g,Q)) - (L_\alpha(f,P) + L_\alpha(g,Q)) < \epsilon.$$

But we have:

$$U_\alpha(f+g, P \cup Q) - L_\alpha(f+g, P \cup Q) \leq \big(U_\alpha(f,P) + U_\alpha(g,Q)\big) - (L_\alpha(f,P) + L_\alpha(g,Q)) < \epsilon.$$

Thus $f + g \in R_\alpha[a,b]$.

Since $L_\alpha(f,P) + L_\alpha(g,P) \leq L_\alpha(f+g,P)$

$U_\alpha(f,P) + U_\alpha(g,P) \geq U_\alpha(f+g,P)$

for all partitions P:

$$\int_{\_a}^{b} f d\alpha + \int_{\_a}^{b} g d\alpha \leq \int_{\_a}^{b} (f+g) d\alpha$$

$$\leq \int_a^{\bar{b}} (f+g)\, d\alpha$$

$$\leq \int_a^{\bar{b}} f\, d\alpha + \int_a^{\bar{b}} g\, d\alpha.$$

But $\int_a^{\bar{b}} f\, d\alpha = \int_{\_a}^{b} f d\alpha = \int_a^b f d\alpha$ and $\int_a^{\bar{b}} g\, d\alpha = \int_{\_a}^{b} g d\alpha = \int_a^b g d\alpha$, so

$$\int_a^{\bar{b}} (f+g)\, d\alpha = \int_{\_a}^{b} (f+g) d\alpha = \int_a^b (f+g) d\alpha.$$

Thus $\int_a^b (f+g) d\alpha = \int_a^b f d\alpha + \int_a^b g d\alpha$.

iii. If $f \le g$ on $[a,b]$ then for any partition P:

$$m_i(f) \le m_i(g) \quad \text{and} \quad M_i(f) \le M_i(g).$$

Thus: $L_\alpha(f,P) \le L_\alpha(g,P)$ and $U_\alpha(f,P) \le U_\alpha(g,P)$.

Since $f, g \in R_\alpha[a,b]$

$\int_a^{\bar{b}} f\, d\alpha = \int_{\_a}^{b} f d\alpha = \int_a^b f d\alpha$ and $\int_a^{\bar{b}} g\, d\alpha = \int_{\_a}^{b} g d\alpha = \int_a^b g d\alpha$

and $\int_{\_a}^{b} f d\alpha \le \int_{\_a}^{b} g d\alpha$ so $\int_a^b f d\alpha \le \int_a^b g d\alpha$.

iv. Notice that for any real numbers A, B we have:

$$||A| - |B|| \le |A - B|,$$

since if $|A| \ge |B|$, then by the triangle inequality:

$$|A| = |(A - B) + B| \le |A - B| + |B|$$

$$\big||A| - |B|\big| = |A| - |B| \le |A - B|.$$

If $|B| \ge |A|$ then

$$|B| = |(B - A) + A| \le |A - B| + |A|$$

$$\big||B| - |A|\big| = |B| - |A| \le |A - B|$$

$$\big||A| - |B|\big| = |B| - |A| \le |A - B|.$$

Thus for a function f for any $s, t \in [x_{i-1}, x_i]$ we have:

$$||f(s)| - |f(t)|| \le |f(s) - f(t)|.$$

Hence $|M_i(|f|) - m_i(|f|)| \le |M_i(f) - m_i(f)|$.

Thus $U_\alpha(|f|, P) - L_\alpha(|f|, P) \leq U_\alpha(f, P) - L_\alpha(f, P)$.

So since $f \in R_\alpha[a, b]$ there exists a partition P such that

$$U_\alpha(|f|, P) - L_\alpha(|f|, P) \leq U_\alpha(f, P) - L_\alpha(f, P) < \epsilon.$$

Hence $|f| \in R_\alpha[a, b]$.

Since $-f, f \leq |f| \leq \|f\|_\infty$, by i and iii we have:

$$|\int_a^b f d\alpha| \leq \int_a^b |f| d\alpha \leq \int_a^b \|f\|_\infty d\alpha = \|f\|_\infty(\alpha(b) - \alpha(a)).$$

v. To show $fg \in R_\alpha[a, b]$ we start by showing that $f^2 \in R_\alpha[a, b]$.

Notice that $(f(x))^2 - (f(y))^2 = (f(x) + f(y))(f(x) - f(y))$

$$\leq 2\|f\|_\infty |f(x) - f(y)|.$$

Thus:

$$U_\alpha(f^2, P) - L_\alpha(f^2, P) = \sum_{i=1}^{n} M_i(f^2)\big(\Delta\alpha(x_i)\big) - \sum_{i=1}^{n} m_i(f^2)\big(\Delta\alpha(x_i)\big)$$

$$= \sum_{i=1}^{n}(M_i(f^2) - m_i(f^2))\big(\Delta\alpha(x_i)\big)$$

$$\leq \sum_{i=1}^{n} 2\|f\|_\infty(M_i(f) - m_i(f))\big(\Delta\alpha(x_i)\big)$$

$$= 2\|f\|_\infty(U_\alpha(f, P) - L_\alpha(f, P)).$$

Since $f \in R_\alpha[a, b]$ given any $\epsilon > 0$ there exists a partition P such that

$$U_\alpha(f, P) - L_\alpha(f, P) < \epsilon/(2\|f\|_\infty).$$

Hence for this partition:

$$U_\alpha(f^2,P) - L_\alpha(f^2,P) \leq 2\|f\|_\infty(U_\alpha(f,P) - L_\alpha(f,P))$$
$$\leq 2\|f\|_\infty(\frac{\epsilon}{2\|f\|_\infty}) = \epsilon\ .$$

So $f^2 \in R_\alpha[a,b]$.

Now notice that:

$$4fg = (f+g)^2 - (f-g)^2.$$

By ii, since $f, g \in R_\alpha[a,b]$, so are $f \pm g$.

Since $f \in R_\alpha[a,b]$, so are f^2, $(f+g)^2$, and $(f-g)^2$.
Thus $fg \in R_\alpha[a,b]$.

Now to get the Cauchy-Schwarz inequality:

$$|\int_a^b fg d\alpha| \leq (\int_a^b f^2 d\alpha)^{\frac{1}{2}} (\int_a^b g^2 d\alpha)^{\frac{1}{2}},$$

we see that for any $\lambda \in \mathbb{R}$:

$$0 \leq \int_a^b (f+\lambda g)^2\, d\alpha = \int_a^b (f^2 + 2\lambda fg + \lambda^2 g^2)\, d\alpha$$
$$= \int_a^b f^2 d\alpha + 2\lambda \int_a^b fg d\alpha + \lambda^2 \int_a^b g^2 d\alpha\,.$$

This is a nonnegative quadratic in λ. Thus the discriminant

$$b^2 - 4ac \leq 0.$$

Thus we have:

$$\left(2\int_a^b fg\,d\alpha\right)^2 - 4\left(\int_a^b f^2 d\alpha\right)\left(\int_a^b g^2 d\alpha\right) \le 0$$

$$\left(2\int_a^b fg\,d\alpha\right)^2 \le 4\left(\int_a^b f^2 d\alpha\right)\left(\int_a^b g^2 d\alpha\right)$$

$$\left|\int_a^b fg\,d\alpha\right| \le \left(\int_a^b f^2 d\alpha\right)^{\frac{1}{2}}\left(\int_a^b g^2 d\alpha\right)^{\frac{1}{2}}.$$

Ex. Show that $\left|\int_\pi^{2\pi} \frac{sinx}{x} dx\right| \le \frac{1}{2}$.

Apply the Cauchy-Schwarz inequality with $f(x) = sinx$, $g(x) = \frac{1}{x}$:

$$\left|\int_\pi^{2\pi} \frac{sinx}{x} dx\right| \le \sqrt{\int_\pi^{2\pi} \sin^2 x dx}\sqrt{\int_\pi^{2\pi} \frac{1}{x^2} dx}$$

$$= \sqrt{\int_\pi^{2\pi}\left(\frac{1}{2} - \frac{1}{2}cos2x\right)dx}\sqrt{-\frac{1}{x}\Big|_\pi^{2\pi}}$$

$$= \sqrt{\frac{1}{2}x - \frac{1}{4}sin2x\Big|_\pi^{2\pi}}\sqrt{-\frac{1}{2\pi} + \frac{1}{\pi}}$$

$$= \sqrt{\pi - \frac{\pi}{2}}\sqrt{\frac{1}{2\pi}} = \sqrt{\frac{\pi}{2}}\sqrt{\frac{1}{2\pi}} = \frac{1}{2}.$$

Ex. Prove $\lim_{n\to\infty}\int_0^1 (x^n sin\pi x)\,dx = 0.$

By the Cauchy-Schwarz inequality we have:

$$0 \le |\int_0^1 (x^n sin\pi x)\,dx| \le \sqrt{\int_0^1 x^{2n}dx}\sqrt{\int_0^1 (\sin^2 \pi x)dx}$$

$$= \sqrt{\frac{1}{2n+1}x^{2n+1}|_0^1}\sqrt{\int_0^1 \left(\frac{1}{2}-\frac{1}{2}cos2\pi x\right)dx}$$

$$= \sqrt{\frac{1}{2n+1}}\sqrt{(\frac{1}{2}x-\frac{1}{4\pi}sin2\pi x)|_0^1}$$

$$= \sqrt{\frac{1}{2n+1}}\sqrt{\frac{1}{2}} = \sqrt{\frac{1}{4n+2}}$$

Since $\lim_{n\to\infty}\sqrt{\frac{1}{4n+2}} = 0,\ \lim_{n\to\infty}\int_0^1 (x^n sin\pi x)\,dx = 0$ by the squeeze theorem.

Parts i and ii of this theorem tell us that $R_\alpha[a,b]$ is a vector space. (since $f(x) = 0 \in R_\alpha[a,b]$) . There are some obvious candidates for a norm on $R_\alpha[a,b]$.

$$\|f\| = \int_a^b |f(x)|d\alpha$$

$$\|f\| = \sup_{a\le x\le b}|f(x)|.$$

The problem with the first candidate is that it isn't always a norm. That is, it's possible to find a nonconstant increasing function α and a nonzero function $f \in R_\alpha[a,b]$ such that $\int_a^b |f(x)|d\alpha = 0$ (for example, $\alpha(x) = x$ and $f(x) = 1$, if $x = 0$, and $f(x) = 0$ otherwise).

The sup-norm is a good candidate for $R_\alpha[a,b]$ because by the next theorem $R_\alpha[a,b]$ is closed under uniform convergence. That is $R_\alpha[a,b]$ is complete with $\|f\| = \sup_{a\le x\le b} |f(x)|$.

Theorem: Let $\{f_n\}$ be a sequence in $R_\alpha[a,b]$. If $\{f_n\} \to f$ uniformly on $[a,b]$, then $f \in R_\alpha[a,b]$ and $\lim_{n\to\infty} \int_a^b f_n d\alpha = \int_a^b f d\alpha$.

Proof. Let's first show $f \in R_\alpha[a,b]$ by showing given any $\epsilon > 0$, there is a partition, $P = \{x_0, x_1, \dots, x_n\}$, such that

$$U_\alpha(f,P) - L_\alpha(f,P) < \epsilon.$$

$\{f_n\} \to f$ uniformly on $[a,b]$ so given any $\epsilon > 0$, there exists an $N \in \mathbb{Z}^+$ such that if $n \ge N$ then

$$|f_n(x) - f(x)| < \frac{\epsilon}{4(\alpha(b)-\alpha(a))} \quad \text{for all } x \in [a,b].$$

By the triangle inequality we have for all $s, t \in [a,b]$:

$$\begin{aligned} |f(s) - f(t)| &\le |f(s) - f_N(s)| + |f_N(s) - f_N(t)| + |f_N(t) - f(t)| \\ &\le \frac{\epsilon}{4(\alpha(b)-\alpha(a))} + |\, f_N(s) - f_N(t)| + \frac{\epsilon}{4(\alpha(b)-\alpha(a))} \\ &= |f_N(s) - f_N(t)| + \frac{\epsilon}{2(\alpha(b)-\alpha(a))}. \end{aligned}$$

Since $f_N \in R_\alpha[a,b]$ we know that given any $\epsilon > 0$ there exists a partition P such that $U_\alpha(f_N,P) - L_\alpha(f_N,P) < \frac{\epsilon}{2}$.

Then we have:

$$U_\alpha(f,P) - L_\alpha(f,P) = \sum_{i=1}^{n}(M_i(f) - m_i(f))\big(\Delta\alpha(x_i)\big)$$

$$\leq \sum_{i=1}^{n}(M_i(f_N) - m_i(f_N) + \tfrac{\epsilon}{2(\alpha(b)-\alpha(a))})\big(\Delta\alpha(x_i)\big)$$

$$= \sum_{i=1}^{n}(M_i(f_N) - m_i(f_N))\big(\Delta\alpha(x_i)\big) + \sum_{i=1}^{n}(\tfrac{\epsilon}{2(\alpha(b)-\alpha(a))})\big(\Delta\alpha(x_i)\big)$$

$$= U_\alpha(f_N,P) - L_\alpha(f_N,P) + (\tfrac{\epsilon}{2(\alpha(b)-\alpha(a))})\big(\alpha(b) - \alpha(a)\big)$$

$$< \quad \frac{\epsilon}{2} \quad + \quad \frac{\epsilon}{2} \quad = \epsilon.$$

So $f \in R_\alpha[a,b]$.

To see $\lim_{n\to\infty} \int_a^b f_n d\alpha = \int_a^b f d\alpha$, notice that:

$$0 \leq |\int_a^b (f_n - f) d\alpha| \leq \int_a^b |f_n - f| d\alpha \leq \|f_n - f\|_\infty[\alpha(b) - \alpha(a)]$$

and the RHS goes to 0 as $n \to \infty$.

Thus $R_\alpha[a,b]$ is a Banach space with the sup-norm. Notice also that $C[a,b] \subseteq R_\alpha[a,b]$ is a closed vector subspace.

Theorem: Let α be continuous and increasing. Given $f \in R_\alpha[a,b]$ and $\epsilon > 0$, there exists

i. A step function h on $[a,b]$ with $\|h\|_\infty \leq \|f\|_\infty$ such that

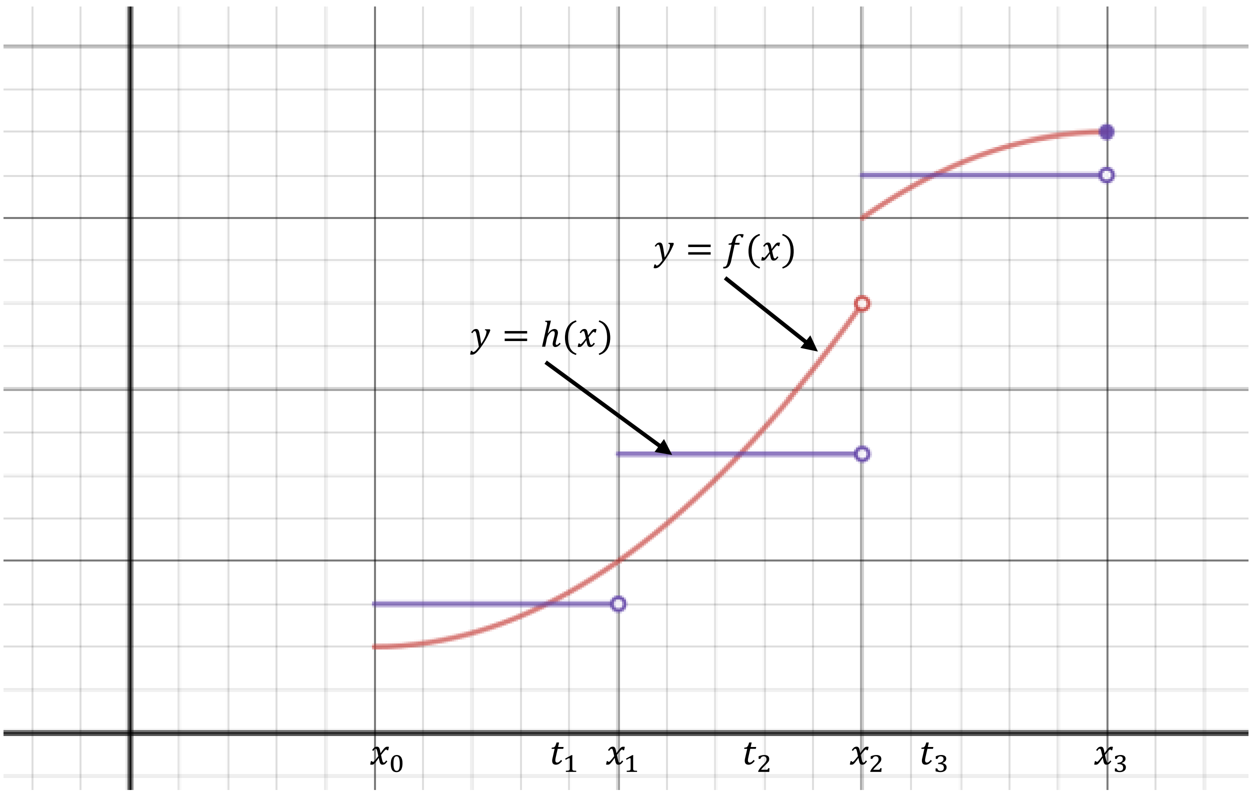
$$\int_a^b |f - h| d\alpha < \epsilon, \quad \text{and}$$

ii. A continuous function g on $[a,b]$ with $\|g\|_\infty \leq \|f\|_\infty$ such that

$$\int_a^b |f - g| d\alpha < \epsilon.$$

Proof. i. Since $f \in R_\alpha[a,b]$, there exists a partition $P = \{x_0, x_1, \dots, x_n\}$, such that: $$U_\alpha(f,P) - L_\alpha(f,P) < \epsilon.$$

For each $i = 1,2,\dots,n$ choose $t_i \in [x_{i-1}, x_i)$ and define a step function by:
$$\begin{aligned} h(x) &= f(t_i) \quad if \quad x \in [x_{i-1}, x_i) \\ &= f(x_n) \quad if \quad x = x_n. \end{aligned}$$



Thus we have $\|h\|_\infty \le \|f\|_\infty$.

Since α is continuous $h \in R_\alpha[a,b]$.

From $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$, for any $c \in [a,b]$
we have:
$$\begin{aligned} \int_a^b |f - h| d\alpha &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f(x) - f(t_i)| d\alpha \\ &\le \sum_{i=1}^n (M_i - m_i)\Delta\alpha_i \\ &= U_\alpha(f,P) - L_\alpha(f,P) < \epsilon. \end{aligned}$$

ii Since α is continuous on $[a, b]$, it is also uniformly continuous. Thus given $\epsilon > 0$ we can always choose a $\delta > 0$ such that :

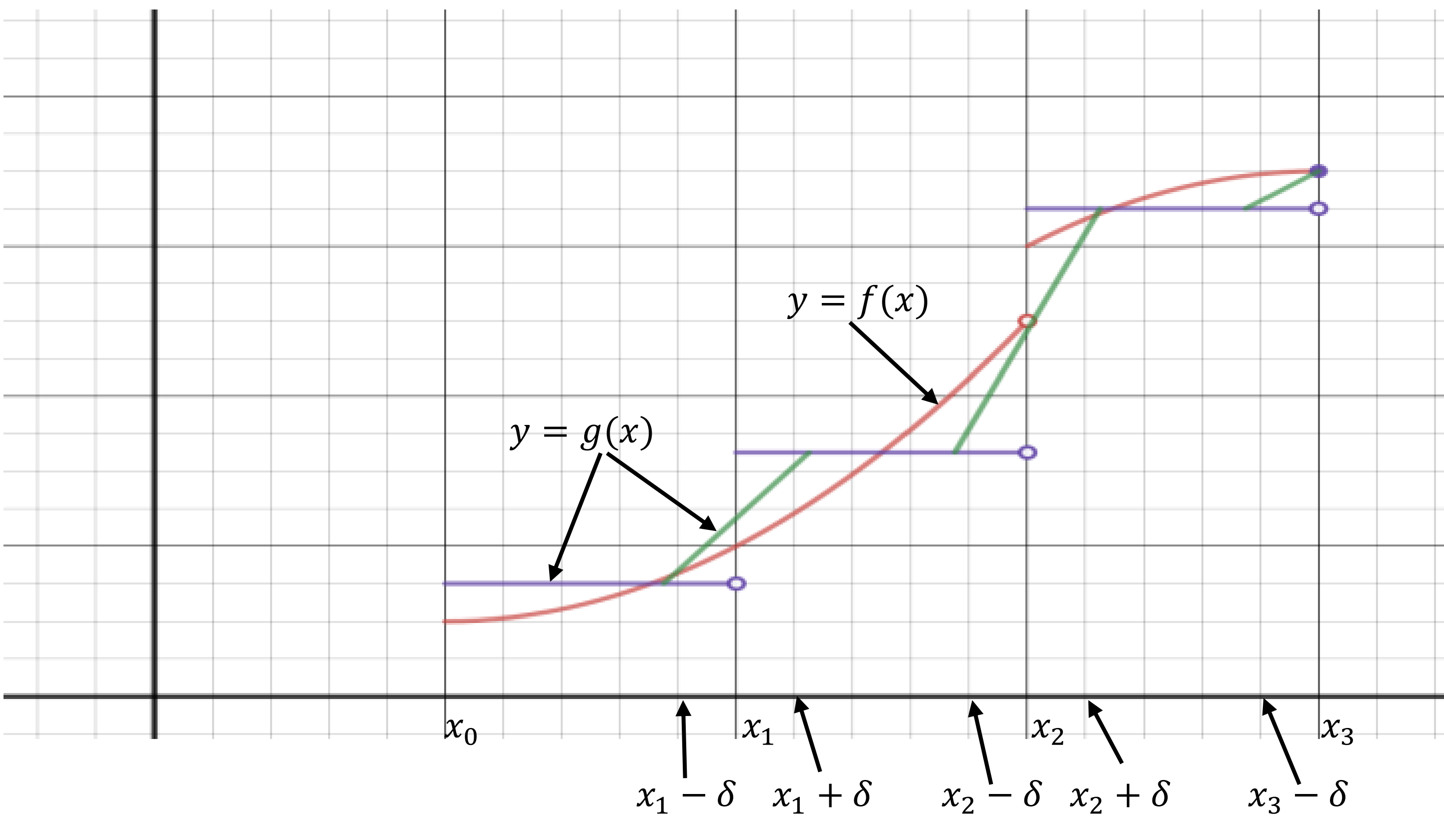
$$\delta < \min\left(\frac{\Delta x_i}{2}\middle|\ i = 1,2, \dots, n\right) \text{ such that } |\alpha(x) - \alpha(y)| < \frac{\epsilon}{n+1}$$

for x, y in $[x_i - \delta, x_i + \delta] \cap [a, b]$.

Now let $g(x)$ be a polygonal function that agrees with h at each node

$$x_0,\ (x_0 + \delta), (x_1 - \delta), (x_1 + \delta), \dots\ ,\ (x_n - \delta),\ x_n.$$

g is a piecewise linear continuous function that agrees with h on each subinterval $[(x_{i-1} + \delta), (x_i - \delta)]$ and is linear on each interval $[(x_i - \delta), (x_i + \delta)]$.



Then $\|g\|_\infty \leq \|h\|_\infty \leq \|f\|_\infty$, $\quad g \in R_\alpha[a,b]$, and

$$\int_a^b |h-g|d\alpha = \int_{x_0}^{x_0+\delta} |h-g|d\alpha + \sum_{i=1}^{n-1} \int_{x_i-\delta}^{x_i+\delta} |h-g|d\alpha + \int_{x_n-\delta}^{x_n} |h-g|d\alpha$$

$$\leq 2\|f\|_\infty\left(\frac{\epsilon}{n+1}\right) + \sum_{i=1}^{n-1}(2\|f\|_\infty\left(\frac{\epsilon}{n+1}\right)) + 2\|f\|_\infty\left(\frac{\epsilon}{n+1}\right)$$

$$= 2\epsilon\|f\|_\infty.$$

Now using the triangle inequality we get:

$$\int_a^b |f-g|d\alpha \leq \int_a^b |f-h|d\alpha + \int_a^b |h-g|d\alpha < \epsilon + 2\epsilon\|f\|_\infty.$$

(We could have chosen the length of the subintervals so that $\int_a^b |h-g|d\alpha < \frac{\epsilon}{2}$ and h so that $\int_a^b |f-h|d\alpha < \frac{\epsilon}{2}$) .

Ex. Given that $f \in R_\alpha[a,b]$ and $\epsilon > 0$, prove there exists a polynomial, $p(x)$, such that $\int_a^b |f(x)-p(x)|d\alpha < \epsilon$.

From the previous theorem we know that there exists a $g \in C[a,b]$ such that

$$\int_a^b |f(x)-g(x)|d\alpha < \frac{\epsilon}{2}.$$

By the Weierstrass approximation theorem we know there exists a polynomial, $p(x)$, on $[a,b]$ where $\sup_{a\leq x\leq b} |g(x)-p(x)| < \frac{\epsilon}{2(\alpha(b)-\alpha(a))}$.

Thus we have:

$$\begin{aligned}\int_a^b |f(x)-p(x)|d\alpha &\leq \int_a^b |f(x)-g(x)|d\alpha + \int_a^b |g(x)-p(x)|d\alpha \\ &< \frac{\epsilon}{2} + \int_a^b \frac{\epsilon}{2(\alpha(b)-\alpha(a))}d\alpha \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Ex. (Riemann's Lemma) Let $f(x) \in R[-\pi,\pi]$. Prove that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 0.$$

We must show given any $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$ such that if $n \geq N$, then $| \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx - 0 | < \epsilon$.

By the previous theorem we know that there exists a step function, h, on $[-\pi,\pi]$, such that $\int_{-\pi}^{\pi} |f-h| dx < \frac{\epsilon}{2}$.

Let P be a partition of $[-\pi,\pi]$, $P = \{x_0, x_1, \dots, x_m\}$, such that

$h(x) = \sum_{i=1}^{m} c_i \chi_{[x_{i-1},x_i)}$ and $h(x_m) = f(x_m)$.

Let's show that $\lim_{n\to\infty} \int_{-\pi}^{\pi} h(x)\cos(nx)\,dx = 0$.

$$\begin{aligned}
\int_{-\pi}^{\pi} h(x)\cos(nx)\,dx &= \int_{-\pi}^{\pi} (\sum_{i=1}^{m} c_i \chi_{[x_{i-1},x_i)}) \cos(nx)\,dx \\
&= \sum_{i=1}^{m} \int_{-\pi}^{\pi} c_i \chi_{[x_{i-1},x_i)}) \cos(nx)\,dx \\
&= \sum_{i=1}^{m} \int_{x_{i-1}}^{x_i} c_i \cos(nx)\,dx \\
&= \sum_{i=1}^{m} c_i (\frac{\sin(nx)}{n})\Big|_{x_{i-1}}^{x_i} \\
&= \frac{1}{n} \sum_{i=1}^{m} c_i [\sin(nx_i) - \sin(nx_{i-1})].
\end{aligned}$$

Thus we have:

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} h(x)\cos(nx)\,dx = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{m} c_i [\sin(nx_i) - \sin(nx_{i-1})] = 0.$$

That means given any $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$ such that if $n \geq N$, then $\left|\int_{-\pi}^{\pi} h(x)\cos(nx)\,dx - 0\right| < \frac{\epsilon}{2}$.

Using this N, if $n \geq N$ then:

$$\left|\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx - 0\right|$$

$$= \left|\int_{-\pi}^{\pi}[(f(x) - h(x))\cos(nx) + h(x)\cos(nx)]\,dx\right|$$

$$= \left|\int_{-\pi}^{\pi}\big(f(x) - h(x)\big)\cos(nx)dx + \int_{-\pi}^{\pi} h(x)\cos(nx)\,dx\right|$$

$$\leq \left|\int_{-\pi}^{\pi}[(f(x) - h(x))\cos(nx)dx\right| + \left|\int_{-\pi}^{\pi} h(x)\cos(nx)\,dx\right|$$

$$\leq \int_{-\pi}^{\pi}|f(x) - h(x)|dx + \left|\int_{-\pi}^{\pi} h(x)\cos(nx)\,dx\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,.$$

Thus $\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 0$.