The Riemann-Stieltjes Integral

The Riemann-Stieltjes Integral is a generalization of the Riemann Integral.

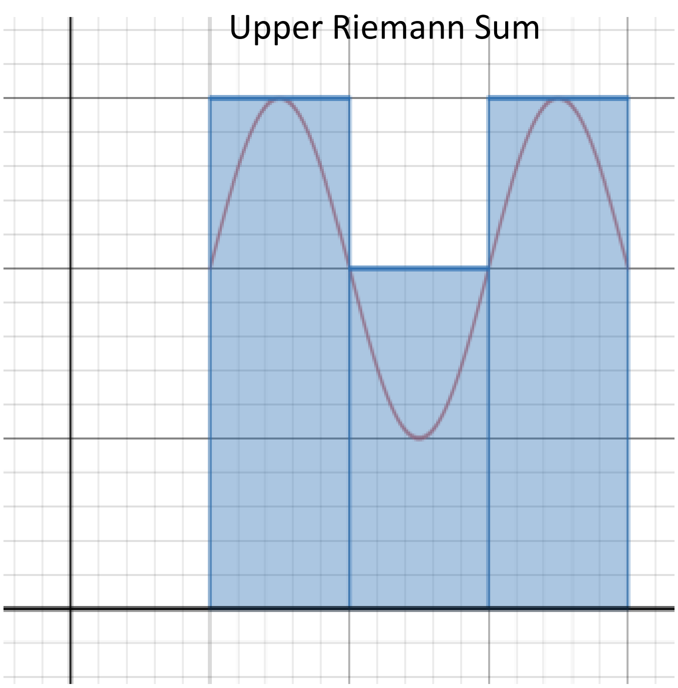
In the Riemann integral of a bounded function f we partition an interval $[a,b]$ into $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. We then consider the upper and lower sums:

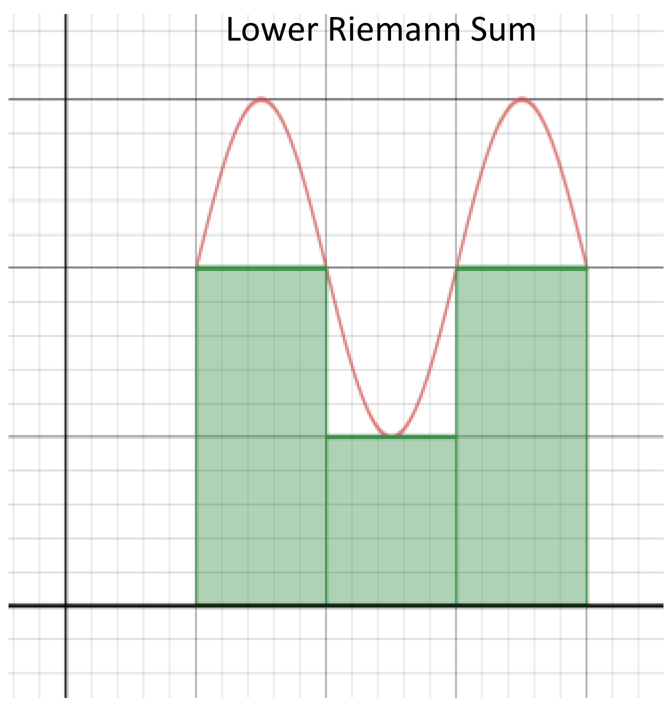
$$U(f,[a,b]) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

$$L(f,[a,b]) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

Where $M_i = \sup\{f(x) |\ x_{i-1} \le x \le x_i\}$

$$m_i = \inf\{f(x) |\ x_{i-1} \le x \le x_i\}.$$





We define the lower and upper Riemann integrals by:

$$\underline{\int}_a^b f\,dx = \sup_P \{L(f,P) |\ P\ a\ partition\ of\ [a,b]\}$$

$$\overline{\int}_a^b f\,dx = \inf_P \{U(f,P) |\ P\ a\ partition\ of\ [a,b]\}.$$

Since f is bounded and $[a, b]$ has finite length, $-\infty < L(f, P) \leq U(f, P) < \infty$ and $-\infty < \int_{\_a}^{b} f dx \leq \int_{a}^{\bar{b}} f dx < \infty$.

If $\int_{\_a}^{b} f dx = \int_{a}^{\bar{b}} f\, dx$ we say that f is Riemann integrable over $[a, b]$ and $\int_{a}^{b} f dx = \int_{\_a}^{b} f dx = \int_{a}^{\bar{b}} f dx$.

For a Riemann-Stieltjes integral of a bounded function f on $[a, b]$, we will start with an increasing function α on $[a, b]$. We then consider the upper and lower sums:

$$U_\alpha(f, [a, b]) = \sum_{i=1}^{n} M_i(\alpha(x_i) - \alpha(x_{i-1}))$$

$$L_\alpha(f, [a, b]) = \sum_{i=1}^{n} m_i(\alpha(x_i) - \alpha(x_{i-1}))$$

where $M_i = \sup\{f(x) |\ x_{i-1} \leq x \leq x_i \}$

$$m_i = \inf\{f(x) |\ x_{i-1} \leq x \leq x_i \}.$$

Notice that $U_\alpha(f, [a, b]) \geq L_\alpha(f, [a, b])$.

We define the lower and upper Riemann-Stieltjes integrals by:

$$\int_{\_a}^{b} f d\alpha = \sup_{P}\{L_\alpha(f, P) |\ P\ a\ partition\ of\ [a, b]\}$$

$$\int_{a}^{\bar{b}} f d\alpha = \inf_{P}\{U_\alpha(f, P) |\ P\ a\ partition\ of\ [a, b]\}.$$

Since f and α are bounded and $[a, b]$ has finite length,

$-\infty < L_\alpha(f, P) \leq U_\alpha(f, P) < \infty$ and $-\infty < \int_{\_a}^{b} f d\alpha \leq \int_{a}^{\bar{b}} f d\alpha < \infty$.

If $\underline{\int}_a^b f d\alpha = \overline{\int}_a^b f\, d\alpha$ we say that f is **Riemann-Stieltjes integrable** over $[a, b]$ and $\int_a^b f d\alpha = \underline{\int}_a^b f d\alpha = \overline{\int}_a^b f d\alpha$.

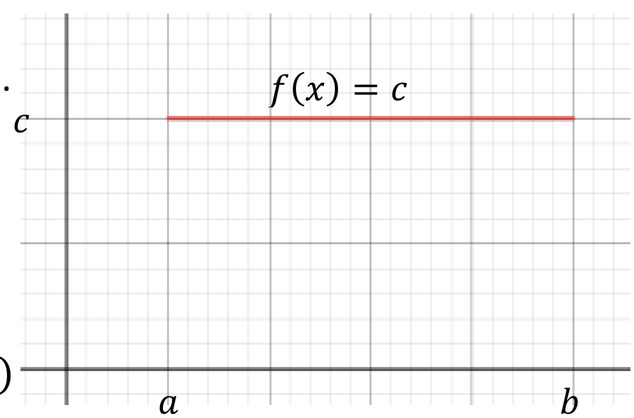
Notice that if: $m = \min(m_1, m_2, \dots, m_n)$

$M = \max(M_1, M_2, \dots, M_n)$

then: $m(\alpha(b) - \alpha(a)) \le \underline{\int}_a^b f d\alpha \le \overline{\int}_a^b f d\alpha \le M(\alpha(b) - \alpha(a))$.

If $\alpha(x) = x$ then we get the usual Riemann integral.

Ex. If $c \in \mathbb{R}$, then $\int_a^b c d\alpha = c(\alpha(b) - \alpha(a))$.



For any partition P we have:

$$U_\alpha(f, [a, b]) = \sum_{i=1}^{n} M_i(\alpha(x_i) - \alpha(x_{i-1}))$$
$$= \sum_{i=1}^{n} c(\alpha(x_i) - \alpha(x_{i-1}))$$
$$= c\big[(\alpha(x_1) - \alpha(x_0)\big) + (\alpha(x_2) - \alpha(x_1)) + \cdots (\alpha(x_n) - \alpha(x_{n-1})]$$
$$= c(\alpha(x_n) - \alpha(x_0)) = c(\alpha(b) - \alpha(a))$$

$$L_\alpha(f, [a, b]) = \sum_{i=1}^{n} m_i(\alpha(x_i) - \alpha(x_{i-1}))$$
$$= \sum_{i=1}^{n} c(\alpha(x_i) - \alpha(x_{i-1}))$$
$$= c\big[(\alpha(x_1) - \alpha(x_0)\big) + (\alpha(x_2) - \alpha(x_1)) + \cdots (\alpha(x_n) - \alpha(x_{n-1})]$$
$$= c(\alpha(x_n) - \alpha(x_0)) = c(\alpha(b) - \alpha(a)).$$

Thus $\underline{\int_a^b} f\,d\alpha = \sup_P L_\alpha(c,\ P) = c(\alpha(b) - \alpha(a))$

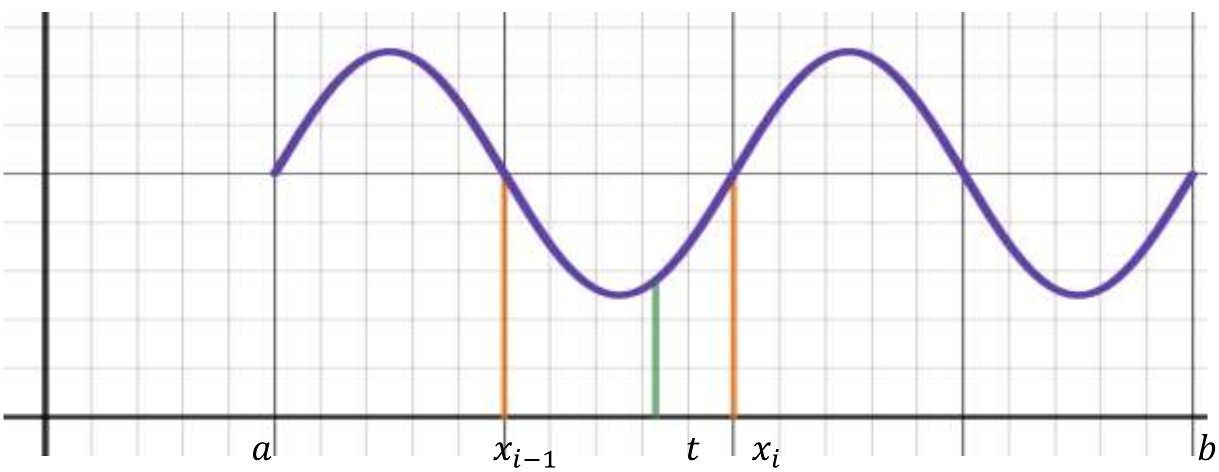
$\overline{\int_a^b} f\,d\alpha = \inf_P U_\alpha(c, P) = c(\alpha(b) - \alpha(a))$

So $\int_a^b f\,d\alpha = c\big(\alpha(b) - \alpha(a)\big)$.

Let P be any partition of $[a, b]$. We can always refine a partition by adding a point (or points).

Proposition: If P' is a refinement of P (i.e. P' contains all of the points of P plus others) then $L_\alpha(f, P') \geq L_\alpha(f, P)$ and $U_\alpha(f, P') \leq U_\alpha(f, P)$.

Proof. Choose any subinterval $x_{i-1} \leq x \leq x_i$ and add a point t.



Let $m_i' = \inf_{x_{i-1} \leq x \leq t} f(x)$ and $m_i'' = \inf_{t \leq x \leq x_i} f(x)$.

Then ${m_i}' \geq m_i$ and ${m_i}'' \geq m_i$.

Now just using the interval $x_{i-1} \leq x \leq x_i$ we have:

$$L_\alpha(f,P') = m_i'\big(\alpha(t) - \alpha(x_{i-1})\big) + m_i''(\alpha(x_i) - \alpha(t))$$
$$\geq m_i\big(\alpha(t) - \alpha(x_{i-1})\big) + m_i(\alpha(x_i) - \alpha(t))$$
$$= m_i(\alpha(x_i) - \alpha(x_{i-1})) = L_\alpha(f,P).$$

A similar argument shows $U_\alpha(f,P') \leq U_\alpha(f,P)$.

Since $\underline{\int_a^b} f d\alpha = \sup_P L_\alpha(f,\ P)$ and $\overline{\int_a^b} f d\alpha = \inf_P U_\alpha(f,P)$

it never "hurts" to refine a partition.

Corollary: For any partitions $P, Q, \quad L_\alpha(f,P) \leq U_\alpha(f,Q)$.

Proof. $L_\alpha(f,P) \leq L_\alpha(f,P \cup Q) \leq U_\alpha(f,P \cup Q) \leq U_\alpha(f,Q)$.

Ex. Suppose $s \in [a,b]$ and $\alpha(x) = 1 \quad if \quad x > s$

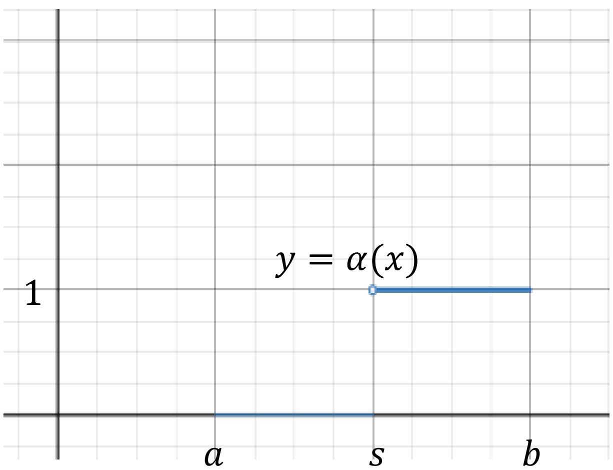
$$= 0 \quad if \quad x \leq s.$$

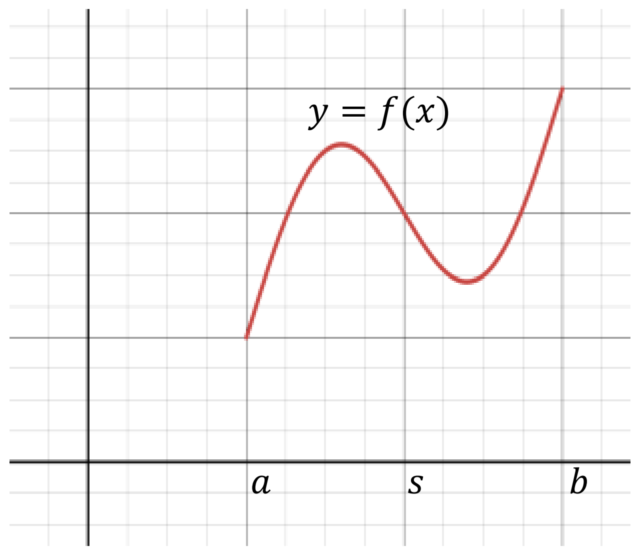
Also assume that $f(x)$ is continuous at $x = s$.

Find $\int_a^b f(x)d\alpha$.

Let P be any partition. If s is not part of P, create a refinement of P that contains s.

Let's say that $x_k = s$. Then notice that $\alpha(x_i) - \alpha(x_{i-1}) = 0$ unless $i = k + 1$, in which case it equals 1.





Thus we have:

$U_\alpha(f,[a,b]) = \sum_{i=1}^{n} M_i(\alpha(x_i) - \alpha(x_{i-1})) = M_{k+1}$

$L_\alpha(f,[a,b]) = \sum_{i=1}^{n} m_i(\alpha(x_i) - \alpha(x_{i-1})) = m_{k+1}$.

Since $f(x)$ is continuous at $x = s$ we know that $\lim_{x \to s^+} f(x) = f(s)$.

Claim: $\lim_{x\to s^+} M_{[s,x]} = f(s)$, and $\lim_{x\to s^+} m_{[s,x]} = f(s)$, where

$$M_{[s,x]} = sup\{f(t)|\ s \le t \le x\ \} \text{ and } m_{[s,x]} = inf\{f(t)|\ s \le t \le x\ \}$$

Since $\lim_{x\to s^+} f(x) = f(s)$ given any $\epsilon > 0$ there exists a $\delta > 0$ such that if $s \le x < s + \delta$ then $|f(s) - f(x)| < \epsilon$.

But this implies that if $s \le x < s + \delta$ then $\sup_{s\le t\le x} |f(s) - f(t)| < \epsilon$.

Thus we have: if $s \le x < s + \delta$ then $\left|f(s) - M_{[s,x]}\right| < \epsilon$.

Similarly, if $s \le x < s + \delta$ then $\inf_{s\le t\le x} |f(s) - f(t)| < \epsilon$ and hence

$\left|f(s) - m_{[s,x]}\right| < \epsilon$.

Thus we have $\lim_{x\to s^+} M_{[s,x]} = f(s)$, and $\lim_{x\to s^+} m_{[s,x]} = f(s)$.

Thus as we take more refinements of P, M_{k+1} and m_{k+1} will converge to $f(s)$.

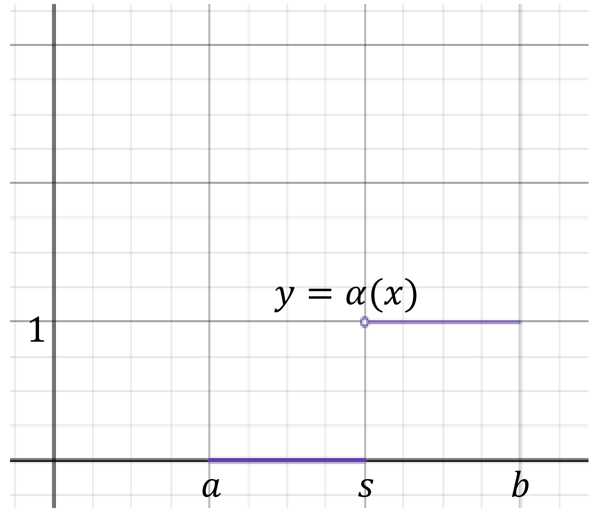
Hence

$$\int_a^{\bar{b}} f d\alpha = \inf_P U_\alpha(f,P) = f(s) \quad \int_{\_a}^{b} f d\alpha = \sup_P L_\alpha(f,\ P) = f(s).$$

Thus $\int_a^b f(x) d\alpha = f(s)$.

Ex. Suppose $s \in [a, b]$ and $\alpha(x) = 1 \quad if \;\; x > s$

$$= 0 \quad if \;\; x \leq s.$$

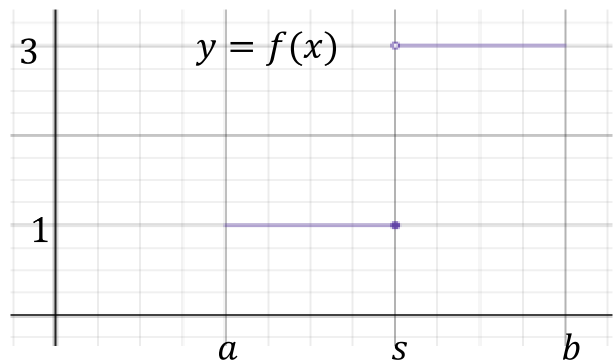


a. Determine if $f(x) = 3 \quad if \;\; x > s$

$$= 1 \quad if \;\; x \leq s$$

is Riemann-Stieltjes integrable on $[a, b]$.

If so find $\int_a^b f(x) d\alpha$.

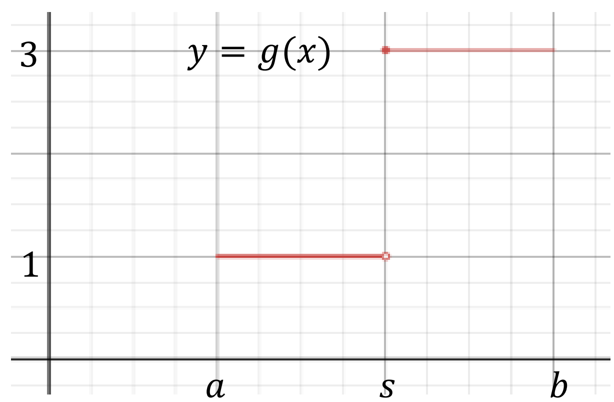


b. Determine if $g(x) = 3 \quad if \;\; x \geq s$

$$= 1 \quad if \;\; x < s$$

is Riemann-Stieltjes integrable on $[a, b]$.

If so find $\int_a^b g(x) d\alpha$.



a. As in the previous example, let P be any partition and let $x_k = s \in P$, then the only non-zero values for the $M_i's$ and $m_i's$ are M_{k+1} and m_{k+1}.

$M_{k+1} = 3,$ so $\overline{\int_a^b} f d\alpha = \inf_P U_\alpha(f, P) = 3$

$m_{k+1} = 1$ so $\underline{\int_a^b} f d\alpha = \sup_P L_\alpha(f, P) = 1$

So f is not Riemann-Stieltjes integrable on $[a, b]$.

b. In this case f is continuous from the right and we have

$M_{k+1} = 3,$ so $\overline{\int_a^b} g d\alpha = \inf_P U_\alpha(g, P) = 3$

$m_{k+1} = 3$ so $\underline{\int_a^b} g d\alpha = \sup_P L_\alpha(g, P) = 3$

Thus g is Riemann-Stieltjes integrable on $[a, b]$ and $\int_a^b g(x) d\alpha = 3$.

Def. $\boldsymbol{R_\alpha[a, b]}$ denotes all bounded functions on $[a, b]$ which are Riemann-Stieltjes integrable with respect to α.

When $\alpha(x) = x$, we write $R[a, b]$ for the space of bounded Riemann integrable functions.

Notice that $R_\alpha[a, b] \subseteq B[a, b]$ =bounded functions on $[a, b]$.

Theorem: Let $\alpha\colon [a, b] \to \mathbb{R}$ be increasing. A bounded function $f\colon [a, b] \to \mathbb{R}$ is in $R_\alpha[a, b]$ if and only if, given $\epsilon > 0$, there exists a partition P of $[a, b]$ such that $U_\alpha(f, P) - L_\alpha(f, P) < \epsilon$.

Proof. Suppose $f \in R_\alpha[a, b]$ and let $I = \int_a^b f(x)d\alpha$. Let's show that there exists a partition P of $[a, b]$ such that $U_\alpha(f, P) - L_\alpha(f, P) < \epsilon$.

Given $\epsilon > 0$, choose partitions P and Q of $[a, b]$ such that

$$I - \frac{\epsilon}{2} < L_\alpha(f, P) \text{ and } U_\alpha(f, Q) < I + \frac{\epsilon}{2}.$$

We know we can do this because

$$\int_a^b f d\alpha = \inf_P U_\alpha(f, P) = \sup_P L_\alpha(f, P).$$

The partition $P' = P \cup Q$ will have:

$$U_\alpha(f, P') \leq U_\alpha(f, Q) < I + \frac{\epsilon}{2} < L_\alpha(f, P) + \epsilon \leq L_\alpha(f, P') + \epsilon.$$

So $U_\alpha(f, P') - L_\alpha(f, P') < \epsilon$.

Now assume for every $\epsilon > 0$ there is a partition P for which $U_\alpha(f, P) - L_\alpha(f, P) < \epsilon$. Let's show that $f \in R_\alpha[a, b]$.

Since: $L_\alpha(f, P) \leq \underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha \leq U_\alpha(f, P)$, for any partition P, we have

$$0 \leq \overline{\int_a^b} f d\alpha - \underline{\int_a^b} f d\alpha \leq U_\alpha(f, P) - L_\alpha(f, P) < \epsilon.$$

Since this is true for any $\epsilon > 0$, $\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha$, and $f \in R_\alpha[a, b]$.

Ex. Let $\alpha(x) = x$ and

$$f(x) = 1 \quad \text{if } x = 0$$
$$= 0 \quad \text{if } x > 0.$$

Show that $f(x) \in R_\alpha[0,1]$.

Let $\epsilon > 0$ be given.

Choose $n \in \mathbb{Z}^+$ such that $\frac{1}{n} < \epsilon$.

Let $P = \{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, 1\}$ be a partition of $[0,1]$.

Notice that for this partition P we have:

$$U_\alpha(f,P) = \sum_{i=1}^{n} M_i\big(\alpha(x_i) - \alpha(x_{i-1})\big)$$
$$= \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$
$$= (x_1 - x_0) = \frac{1}{n}$$

$$L_\alpha(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$
$$= 0.$$

So we have:

$$U_\alpha(f,P) - L_\alpha(f,P) = \frac{1}{n} < \epsilon.$$

Thus $f(x) \in R_\alpha[0,1]$.

Theorem: $C[a,b] \subseteq R_\alpha[a,b]$.

Proof. Let $f \in C[a,b]$ and let $\epsilon > 0$.

Since f is continuous on a closed, bounded interval (a compact set) it is uniformly continuous on $[a,b]$.

Thus we can choose a $\delta > 0$ such that if $|x-y| < \delta$ then

$$|f(x) - f(y)| < \frac{\epsilon}{\alpha(b)-\alpha(a)}.$$

If P is any partition with $x_i - x_{i-1} < \delta$, for all i, then $M_i - m_i < \frac{\epsilon}{\alpha(b)-\alpha(a)}$ for all i and:

$$U_\alpha(f,P) - L_\alpha(f,P) = \sum_{i=1}^n (M_i - m_i)\Delta\alpha_i; \text{ where } \Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

$$< \frac{\epsilon}{\alpha(b)-\alpha(a)} \sum_{i=1}^n \Delta\alpha_i$$

$$= \frac{\epsilon}{\alpha(b)-\alpha(a)} \big(\alpha(b) - \alpha(a)\big) = \epsilon.$$

Thus $f \in R_\alpha[a,b]$.

Ex. Here's an example of a bounded function that is not Riemann-Stieltjes integrable with respect to any nonconstant increasing α.

$$\chi_{\mathbb{Q}}(x) = 1 \quad if\ x \in \mathbb{Q} \cap [a,b]$$
$$= 0 \quad if\ x \notin \mathbb{Q} \cap [a,b].$$

In any partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$, every subinterval $x_{i-1} \le x \le x_i$ will contain both rational and irrational numbers.

Thus $M_i = 1$ and $m_i = 0$ for all i. So we have:

$$L_\alpha(\chi_{\mathbb{Q}}, P) = \sum_{i=1}^{n} m_i(\alpha(x_i) - \alpha(x_{i-1})) = 0.$$

$$U_\alpha(\chi_{\mathbb{Q}}, P) = \sum_{i=1}^{n} M_i(\alpha(x_i) - \alpha(x_{i-1}))$$
$$= \sum_{i=1}^{n}(\alpha(x_i) - \alpha(x_{i-1}))$$
$$= \alpha(b) - \alpha(a) \ne 0$$

Thus $\inf_P U_\alpha(\chi_{\mathbb{Q}}, P) = \alpha(b) - \alpha(a) \ne 0$ (since $\alpha(x)$ is increasing and non-constant)

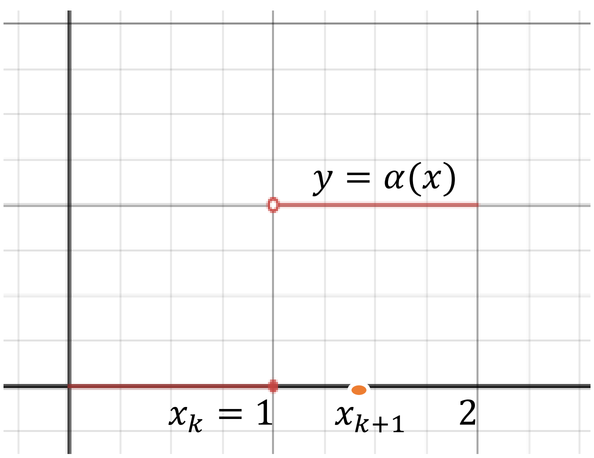
$$\sup_P L_\alpha(\chi_{\mathbb{Q}}, P) = 0.$$

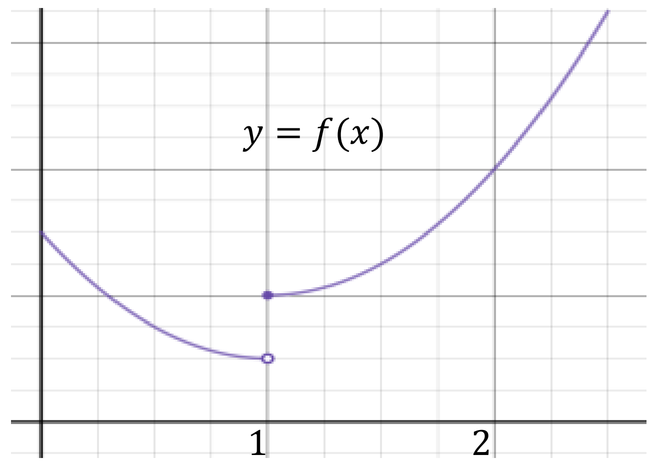
So $\underline{\int}_a^b \chi_{\mathbb{Q}}\, d\alpha \ne \overline{\int}_a^b \chi_{\mathbb{Q}}\, d\alpha$ and $\chi_{\mathbb{Q}}$ is not Riemann-Stieltjes integrable.

Ex. Let $\alpha(x) = \chi_{(1,2]} = 1 \quad$ if $1 < x \leq 2$

$= 0 \quad$ otherwise.

Show that $f(x) \in R_\alpha[0,2]$ if and only if $\lim_{x \to 1^+} f(x) = f(1)$.





We can always choose a partition P of $[0,2]$ that includes the point $x_k = 1$. Then we have:

$$L_\alpha(f, P) = \sum_{i=1}^{n} m_i(\alpha(x_i) - \alpha(x_{i-1})).$$

Since $\alpha(x) = 0$ for $x \leq 1$ and $\alpha(x) = 1$ for $1 < x \leq 2$, then the only nonzero contribution comes from $\alpha(x_{k+1}) - \alpha(x_k) = 1$. Hence

$$L_\alpha(f, P) = m_{k+1}.$$

So we get $\sup_P L_\alpha(f, P) = \lim_{x_{k+1} \to 1^+} m_{k+1}$.

Similarly, $U_\alpha(f, P) = \sum_{i=1}^{n} M_i(\alpha(x_i) - \alpha(x_{i-1})) = M_{k+1}$ and

$\inf_P U_\alpha(f, P) = \lim_{x_{k+1} \to 1^+} M_{k+1}$.

So the only way for $\inf_P U_\alpha(f,P) = \sup_P L_\alpha(f,P)$ is for

$\lim_{x_{k+1}\to 1^+} m_{k+1} = \lim_{x_{k+1}\to 1^+} M_{k+1}$. That is, f is integrable if and only if $\lim_{x_{k+1}\to 1^+} m_{k+1} = \lim_{x_{k+1}\to 1^+} M_{k+1}$.

Claim: $\lim_{x_{k+1}\to 1^+} m_{k+1} = \lim_{x_{k+1}\to 1^+} M_{k+1}$ if and only if $\lim_{x\to 1^+} f(x) = f(1)$.

Assume $\lim_{x_{k+1}\to 1^+} m_{k+1} = \lim_{x_{k+1}\to 1^+} M_{k+1}$, i.e., $\lim_{x_{k+1}\to 1^+}(M_{k+1} - m_{k+1}) = 0$.

Let's show that $\lim_{x\to 1^+} f(x) = f(1)$.

We must show that for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $1 < x < 1+\delta$ then $|f(x) - f(1)| < \epsilon$.

Let $M_{[1,x]} = \sup_{1\le t\le x} f(t), \qquad m_{[1,x]} = \inf_{1\le t\le x} f(t).$

Since $\lim_{x_{k+1}\to 1^+}(M_{k+1} - m_{k+1}) = 0$, we know given $\epsilon > 0$ there exists a $\delta' > 0$ such that if $1 < x < 1+\delta'$ then $\left|M_{[1,x]} - m_{[1,x]}\right| < \epsilon$.

But $|f(x) - f(1)| \le \left|M_{[1,x]} - m_{[1,x]}\right| < \epsilon$.

So if we choose $\delta = \delta'$ then we ensure that $|f(x) - f(1)| < \epsilon$

and $\lim_{x\to 1^+} f(x) = f(1)$.

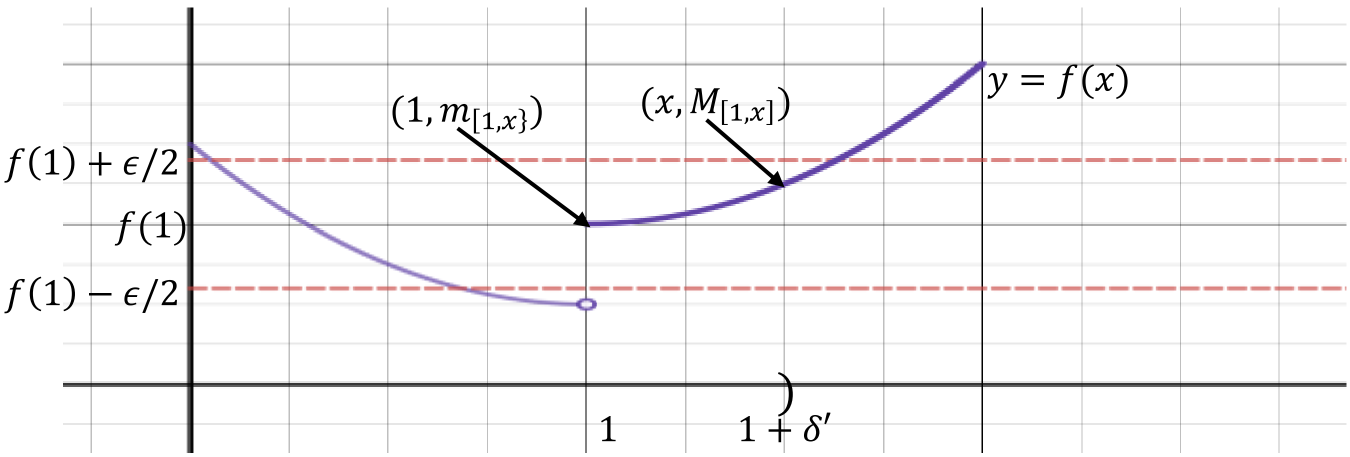
Now assume $\lim_{x \to 1^+} f(x) = f(1)$ and let's show that $\lim_{x_{k+1} \to 1^+} (M_{k+1} - m_{k+1}) = 0.$

We must show that for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $1 < x < 1 + \delta$ then $|M_{[1,x]} - m_{[1,x]}| < \epsilon.$

Since $\lim_{x \to 1^+} f(x) = f(1)$ we know that given any $\epsilon > 0$ there exists a $\delta' > 0$ such that if $1 < x < 1 + \delta'$ then $|f(x) - f(1)| < \frac{\epsilon}{2}.$

Notice that this means that if $1 < x < 1 + \delta'$

$$|M_{[1,x]} - f(1)| < \frac{\epsilon}{2} \quad \text{and} \quad |f(1) - m_{[1,x]}| < \frac{\epsilon}{2}.$$



So choose $\delta = \delta'$ then we have:

$$\begin{aligned} |M_{[1,x]} - m_{[1,x]}| &\leq |M_{[1,x]} - f(1)| + |f(1) - m_{[1,x]}| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

So $\lim_{x_{k+1} \to 1^+} (M_{k+1} - m_{k+1}) = 0.$

Thus $f(x) \in R_\alpha[0,2]$ if and only if $\lim_{x \to 1^+} f(x) = f(1).$