# The Riemann-Stieltjes Integral- HW Problems

1. Show that if $f, g \in R_\alpha[a,b]$, where $\alpha$ is an increasing function and $f(x) \leq g(x)$ for all $x \in [a,b]$ then $\int_a^b f d\alpha \leq \int_a^b g d\alpha$ .

2. Suppose $f^2 \in R_\alpha[a,b]$. Does this imply $f \in R_\alpha[a,b]$? Explain your answer.

3a. Suppose $\alpha$ is continuous, increasing and non-constant on $[0,2]$. From the definition of the Riemann-Stieltjes integral show that

$$f(x) = 1 \quad \text{at } x = 1$$
$$= 0 \quad \text{otherwise}$$

Is Riemann-Stieltjes integrable and $\int_0^2 f d\alpha = 0$.

b. Suppose $f(x) = 1 \quad \text{if } x \in \mathbb{Q}$

$$= -1 \quad \text{if } x \notin \mathbb{Q}.$$

Show that $f$ is not Riemann-Stieltjes integrable with respect to $\alpha$ on $[0,2]$.

4. Let $\alpha(x) = 0 \quad$ if $-1 \leq x < 0$

$= 1 \quad$ if $\ 0 \leq x \leq 1.$

a. Prove that $f(x) = 3 \quad$ if $-1 \leq x \leq 0$

$= 5 \quad$ if $\ 0 < x \leq 1$

is Riemann-Stieltjes integrable on $[-1,1]$ and find $\int_{-1}^{1} f d\alpha$.

b. Prove that $f(x) = 3 \quad$ if $-1 \leq x < 0$

$= 5 \quad$ if $\ 0 \leq x \leq 1$

is not Riemann-Stieltjes integrable on $[-1,1]$.

5. Let $\alpha(x) = 0 \quad$ if $0 \leq x \leq 1$

$= 2 \quad$ if $\ 1 < x \leq 2.$

a. Prove that $f(x) = 4 \quad$ if $\ 0 \leq x \leq 1$

$= 7 \quad$ if $\ 1 < x \leq 2$

is not Riemann-Stieltjes integrable on $[-1,1]$.

b. Prove that $f(x) = 4 \quad$ if $\ 0 \leq x < 1$

$= 7 \quad$ if $\ 1 \leq x \leq 2$

is Riemann-Stieltjes integrable on $[-1,1]$ and find $\int_{0}^{2} f d\alpha$.

6. Let $\alpha(x) = 0$ if $-1 \leq x \leq 0$

$= 1$ if $0 < x \leq 1.$

Prove that if $f(x)$ is continuous on $[-1,1]$ then $f \in R_\alpha[-1,1]$. In that case find $\int_{-1}^{1} f d\alpha$.

Note: $f$ only needs to be continuous from the right at $x = 0$ for the statement to be true.

7. Suppose $f(x)$ is a continuous real-valued function on $\mathbb{R}$. Let $[x]$=the greatest integer less than or equal to $x$. Evaluate

a. $\int_{1}^{n} f d[x]$ if $n \in \mathbb{Z}^{+}$.

b. $\int_{1}^{s} f d[x]$ if $s \in \mathbb{R}$, $s > 1$.

8. Suppose that $f \in R_\alpha[a, b]$, $\alpha$ an increasing function on $[a, b]$. Show that if $[c, d] \subseteq [a, b]$ then $f \in R_\alpha[c, d]$ and $\int_{a}^{b} f d\alpha = \int_{a}^{c} f d\alpha + \int_{c}^{b} f d\alpha$.