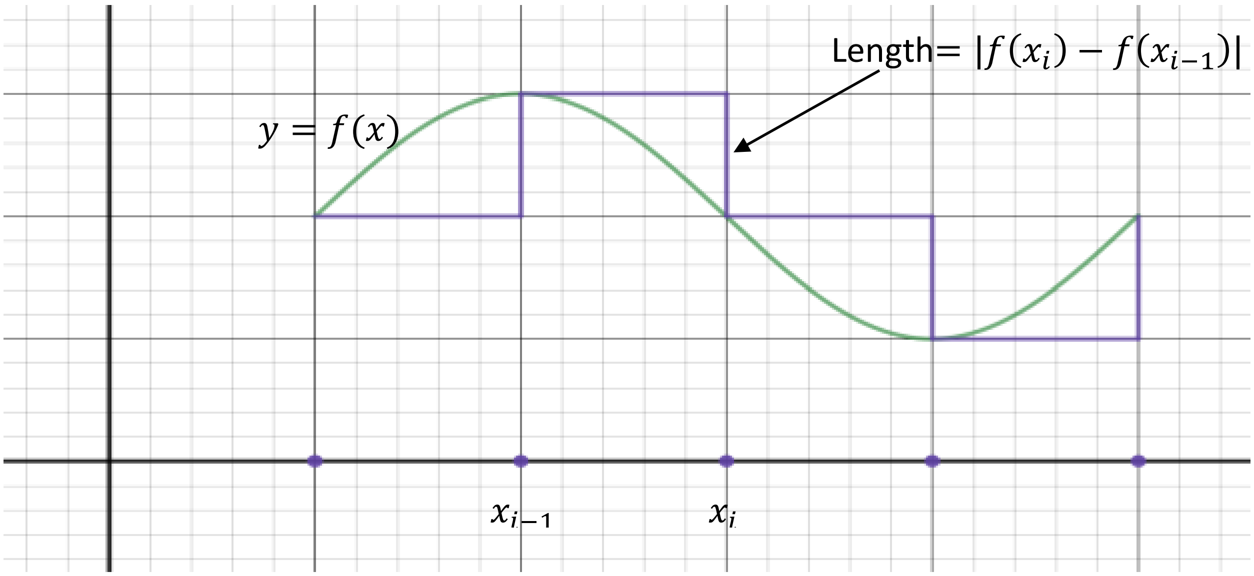
Functions of Bounded Variation: Jordan's Theorem

Def. Let f be a real valued function defined on a closed, bounded interval $[a, b]$ and P a partition $\{x_0, x_1, x_2, \dots, x_k\}$ of $[a, b]$. The **variation of f with respect to P** is defined as:

$$V(f, P) = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|.$$



Def. The **total variation of f** on $[a, b]$ is defined as:

$$TV(f) = \sup\{V(f, P) |\ P\ a\ partition\ of\ [a, b]\}.$$

Def. A real valued function f on the closed, bounded interval $[a, b]$ is said to be of **bounded variation** if $TV(f) < \infty$.

Ex. If f is an increasing function on $[a, b]$, then f is of bounded variation and
$TV(f) = f(b) - f(a)$.

Given any partition P of $[a, b]$: $V(f, P) = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|$

Since f is increasing $f(x_i) - f(x_{i-1}) \geq 0$
so $|f(x_i) - f(x_{i-1})| = f(x_i) - f(x_{i-1})$.

$$\begin{aligned} V(f, P) &= \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \\ &= \big(f(x_1) - f(x_0)\big) + \big(f(x_2) - f(x_1)\big) + \cdots (f(x_k) - f(x_{k-1})) \\ &= f(x_k) - f(x_0) = f(b) - f(a). \end{aligned}$$

Thus $TV(f) = \sup_P V(f, P) = f(b) - f(a)$.

Def. A real valued function on $[a, b]$ is called **Lipschitz** if there exists a $c \in \mathbb{R}$ such that

$$|f(x) - f(y)| \leq c|x - y|, \text{ for all } x, y \in [a, b].$$

Notice that any Lipschitz function is uniformly continuous on $[a, b]$.

We can see this by choosing $\delta = \frac{\epsilon}{c}$.

Thus if $|x - y| < \delta = \frac{\epsilon}{c}$ then

$$|f(x) - f(y)| \leq c|x - y| < c\delta = c\left(\frac{\epsilon}{c}\right) = \epsilon.$$

In addition, if $f(x)$ is differentiable for all $x \in [a, b]$, with $|f'(x)| \leq k$, for some nonnegative real number k, then $f(x)$ is Lipschitz.

This follows from the Mean Value Theorem, since for any $x, y \in [a, b]$:

$$\frac{f(x)-f(y)}{x-y} = f'(c) \text{ for some } c \in (a, b).$$

Thus $\quad \left|\frac{f(x)-f(y)}{x-y}\right| = |f'(c)| \leq k.$

Hence $|f(x) - f(y)| \leq k|x - y|$, for all $x, y \in [a, b]$.

Note: $f(x)$ can be Lipschitz without being differentiable. For example $f(x) = |x|$ is Lipschitz on $[-1,1]$ but not differentiable at $x = 0$.

Ex. Let f be a Lipschitz function on $[a, b]$. Then f is of bounded variation on $[a, b]$ and $TV(f) \leq c(b - a)$ where c is the Lipschitz constant, $|f(u) - f(v)| \leq c|u - v|$ for all $u, v \in [a, b]$.

Let $P = \{x_0, x_1, x_2, \dots, x_k\}$ be any partition of $[a, b]$. Then:

$$V(f, P) = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^{k} c|x_i - x_{i-1}| = c|b - a|.$$

Thus $c|b - a|$ is an upper bound for $V(f, P)$ and $TV(f) \leq c(b - a)$.

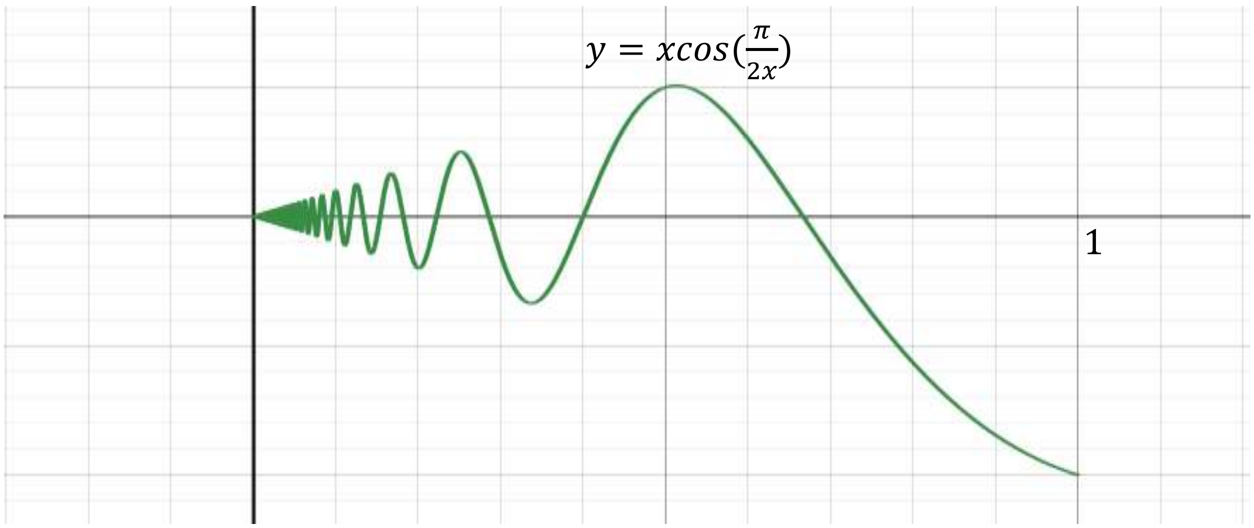
Note: Bounded variation does not imply a function is Lipschitz. For example, $f(x) = \sqrt{x},\ \ 0 < x \leq 1$, is of bounded variation but is not Lipschitz since its derivative is unbounded.

Ex. Define f on $[0,1]$ by

$$f(x) = xcos(\frac{\pi}{2x}) \quad \text{if } 0 < x \leq 1$$

$$= 0 \quad \text{if } x = 0.$$

f is continuous on $[0,1]$, and therefore bounded, but does not have bounded variation.



If we take the partition: $P_n = \{0, \frac{1}{2n}, \frac{1}{2n-1}, \frac{1}{2n-2}, \ldots, \frac{1}{3}, \frac{1}{2}, 1\}$ of $[0,1]$

$f(x_0) = 0$

$$f(x_1) = \frac{1}{2n}\cos\left(\frac{\pi}{2\left(\frac{1}{2n}\right)}\right) = \frac{1}{2n}\cos(n\pi) = \pm\frac{1}{2n}$$

$$f(x_2) = \frac{1}{2n-1}\cos\left(\frac{\pi}{2\left(\frac{1}{2n-1}\right)}\right) = \frac{1}{2n-1}\cos\left(\frac{2n-1}{2}\pi\right) = 0$$

$$f(x_3) = \frac{1}{2n-2}\cos\left(\frac{\pi}{2\left(\frac{1}{2n-2}\right)}\right) = \frac{1}{2n-2}\cos\left(\frac{2n-2}{2}\pi\right) = \pm\frac{1}{2n-2}$$

$$f(x_4) = \frac{1}{2n-3}\cos\left(\frac{2n-3}{2}\pi\right) = 0$$

$\vdots$

So $|f(x_1) - f(x_0)| = \dfrac{1}{2n}$

$|f(x_2) - f(x_1)| = \dfrac{1}{2n}$

$|f(x_3) - f(x_2)| = \frac{1}{2n-2}$

$|f(x_4) - f(x_3)| = \frac{1}{2n-2}$; etc.

Thus $V(f, P_n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$; which diverges as n goes to ∞.

So f is not of bounded variation.

Ex. Notice that if f is $C^1(a, b)$ (i.e. $f'(x)$ is continuous on (a, b)) and continous on $[a, b]$, then for any partition $P = \{x_0, x_1, x_2, \ldots, x_k\}$:

$$f(x_i) - f(x_{i-1}) = \int_{x_{i-1}}^{x_i} f'$$

by the fundamental theorem of Calculus.

Thus we have:

$$|f(x_i) - f(x_{i-1})| = |\int_{x_{i-1}}^{x_i} f'| \leq \int_{x_{i-1}}^{x_i} |f'|.$$

So $$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq \int_a^b |f'|.$$

Thus $TV(f) \leq \int_a^b |f'|$, and f is of bounded variation as long as $\int_a^b |f'| < \infty$.

Ex. All polynomials are of bounded variation on $[a, b]$.

Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be a polynomial. $f(x) \in C^1(a, b)$.

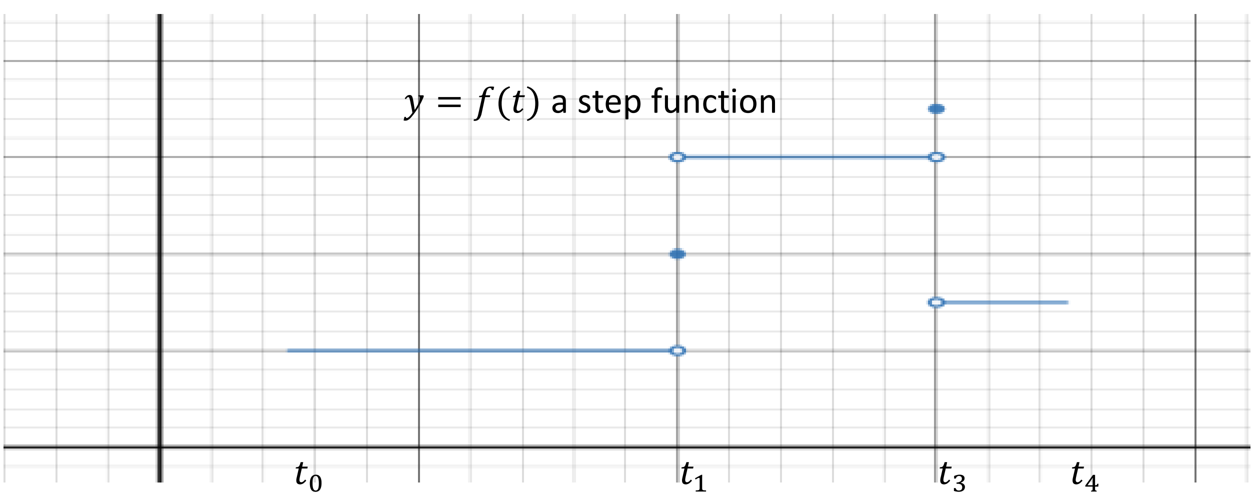
$$f'(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1}.$$

$$\begin{aligned}\int_a^b |a_1 + 2a_2x + \cdots + na_nx^{n-1}|dx \\ &\leq \int_a^b (|a_1| + 2|a_2||x| + \cdots n|a_n||x^{n-1}|)dx \\ &\leq (|a_1| + 2|a_2|(b-a) + \cdots + n|a_n|(b-a)^{n-1})(b-a) \\ &< \infty.\end{aligned}$$

Thus $\int_a^b |f'|dx < \infty$ and $f(x)$ is of bounded variation.

One can also prove this by showing $f'(x)$ is bounded and thus $f(x)$ is Lipschitz.

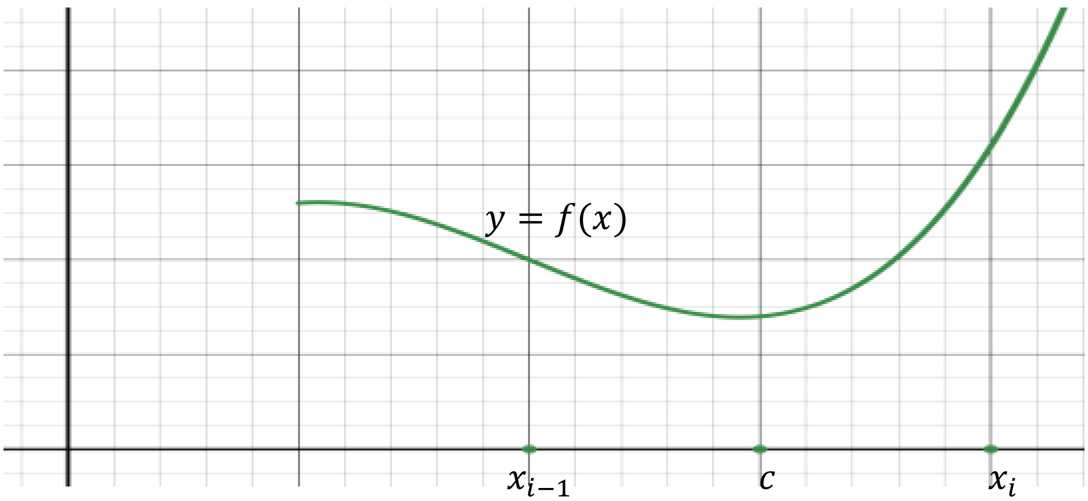
Def. A function $f: [a, b] \to \mathbb{R}$ is called a **step function** if there are finitely many points $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ such that f is constant on each open interval (t_i, t_{i+1}) and f takes on any real values at the t_i's.



Ex. Every step function is of bounded variation. $TV(f)$ is equal to the sum of all left and right hand "jumps" in the graph of f. That is:

$$TV(f) = \sum_{i=0}^{n}(|f(t_i) - \lim_{t \to t_i^+} f(t)| + |f(t_i) - \lim_{t \to t_i^-} f(t)|).$$

Notice if $c \in [a, b]$ and c is not one of the endpoints of a partition P, we can create a refinement P' of P by adding c.



Then by the triangle inequality: $V(f, P) \leq V(f, P')$. Here's why.

The triangle inequality says $|a + b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$.

$$|f(x_i) - f(x_{i-1})| \leq |f(x_i) - f(c)| + |f(c) - f(x_{i-1})|$$

where $a = f(x_i) - f(c)$, $\quad b = f(c) - f(x_{i-1})$

$$a + b = f(x_i) - f(x_{i-1}).$$

Lemma: If $f: [a, b] \to \mathbb{R}$ is of bounded variation, then f is also bounded (i.e. there exists an $M \in \mathbb{R}^+$ such that $|f(x)| \leq M$ for all $x \in [a, b]$) and satisfies: $\|f\|_\infty \leq |f(a)| + TV(f)$.

Proof. Let $a \leq x \leq b$, and set $P = \{a, x, b\}$.

Then $|f(x) - f(a)| \leq V(f, P) \leq TV(f)$.

By the triangle inequality

$$|f(x)| = \left|\left(f(x) - f(a)\right) + f(a)\right| \leq |f(x) - f(a)| + |f(a)|.$$

So $\quad |f(x)| \leq TV(f) + |f(a)|$

$$\Rightarrow \sup_{a \leq x \leq b} |f(x)| = \|f(x)\|_\infty \leq |f(a)| + TV(f).$$

Def. $\boldsymbol{BV[a, b]}$ = {the set of functions of bounded variation on $[a, b]$}.

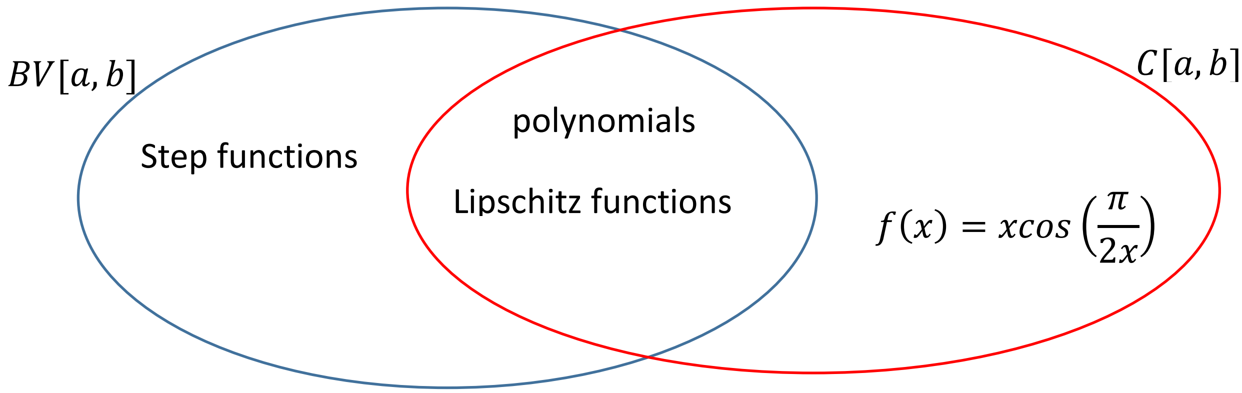
$BV[a, b]$ is a vector space since if $f, g \in BV[a, b]$ then $(af + bg) \in BV[a, b]$ for and $a, b \in \mathbb{R}$ (we will see this shortly), and $f(x) = 0 \in BV[a, b]$.

Notice that $BV[a, b]$ contains some subsets that are dense in $C[a, b]$. For example, all polynomials on $[a, b]$ are of bounded variation and all

polygonal functions on $[a, b]$ are of bounded variation. Thus the closure of $BV[a, b]$ under the sup-norm (i.e. $\|f(x)\|_\infty = \sup_{a \leq x \leq b} |f(x)|$) contains $C[a, b]$.

But we saw that
$$f(x) = \begin{cases} x\cos\left(\frac{\pi}{2x}\right) & if \quad 0 < x \leq 1 \\ 0 & if \quad x = 0 \end{cases}$$

is continuous on $[0,1]$, but not of bounded variation.



Thus there exists a sequence of functions of bounded variation, for example a sequence of polynomials, that converges to $f(x)$ with the sup-norm. Thus the normed linear space $BV[a,b]$ with $\|f(x)\|_\infty = \sup_{a\le x\le b} |f(x)|$, is not complete (and thus not a Banach space with this norm). so if $BV[a,b]$ is going to be a Banach space, we will need to find a different norm. The $TV(f)$ almost works.

Lemma: Let $f, g \in BV[a,b]$ and $c \in \mathbb{R}$. Then

a. $TV(f) = 0$ if and only if f is constant.
b. $TV(cf) = |c|TV(f)$
c. $TV(f+g) \le TV(f) + TV(g)$
d. $TV(|f|) \le TV(f)$
e. $TV\left(f_{[a,b]}\right) = TV\left(f_{[a,c]}\right) + TV\left(f_{[c,b]}\right)$, for $a \le c \le b$.

Proof of c.

Let P be any partition of $[a,b]$, $P = \{x_0, x_1, \dots, x_n\}$.

$$V(f+g,P) = \sum_{i=1}^{n} |(f+g)(x_i) - (f+g)(x_{i-1})|$$
$$\leq \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|$$
$$= V(f,P) + V(g,P).$$

Thus $\sup\limits_{P} V(f+g,P) \leq \sup\limits_{P} V(f,P) + \sup\limits_{P} V(g,P)$.

Hence $TV(f+g) \leq TV(f) + TV(g)$.

$TV(f)$ is not a norm on $BV[a,b]$ because $TV\left(f_{[a,b]}\right) = 0$ does not imply that $f(x) = 0$ (only that $f(x) =$ constant). However, we can create a norm on $BV[a,b]$ by

$$\|f\|_{BV} = |f(a)| + TV\left(f_{[a,b]}\right).$$

From an earlier lemma we had

$$\|f\|_{\infty} \leq |f(a)| + TV\left(f_{[a,b]}\right) = \|f\|_{BV}.$$

Thus convergence in the BV norm, $\|f\|_{BV}$, implies convergence in the uniform convergence norm, $\|f\|_{\infty}$. Let's see why.

We say $f_n \to f$ in the BV norm if given any $\epsilon > 0$, there exists an $N \in \mathbb{Z}^+$ such that if $n \geq N$ then $\|f_n - f\|_{BV} < \epsilon$.

But then $\|f_n - f\|_{\infty} \leq \|f_n - f\|_{BV} < \epsilon$.

Thus the same N that forces $\|f_n - f\|_{BV} < \epsilon$ will also force $\|f_n - f\|_{\infty} < \epsilon$. So $f_n \to f$ in the sup-norm.

Theorem: $BV[a,b]$ is complete under $\|f\|_{BV} = |f(a)| + TV\left(f_{[a,b]}\right)$.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $BV[a,b]$.

Thus $\{f_n\}$ is also a Cauchy sequence under the uniform norm, $\|f\|_\infty$.

That means for all $\epsilon > 0$, there exists an $N \in \mathbb{Z}^+$ such that if

$n, m \geq N$ then $\sup_{a\leq x\leq b} |f_n(x) - f_m(x)| < \epsilon$.

Thus for each $x \in [a,b]$, the sequence of real number$\{f_n(x)\}$

is a Cauchy sequence and so converges to a real number, $f(x)$.

Now let's show that $f(x)$ is first bounded and then of bounded variation.

Each f_n is of bounded variation and thus must be bounded on $[a,b]$.

Since for $m, n \geq N$ we have $\sup_{a\leq x\leq b} |f_n(x) - f_m(x)| < \epsilon$.

if $|f_N(x)| \leq K$, then $|f_n(x)| \leq K + \epsilon$, for all $n \geq N$.

Hence $|f(x)| \leq K + \epsilon$, and f must be bounded on $[a,b]$.

Now we need to show that $f(x) \in BV[a,b]$.

Let P be any partition of $[a,b]$ and $\epsilon > 0$.

Choose N such that if $m, n \geq N$ then $\|f_n - f_m\|_{BV} < \epsilon$.

If $f_n \to f$ pointwise on $[a,b]$, then $V(f_n, P) \to V(f, P)$ for any partition P (this is a HW problem).

Thus for any $n \geq N$:

$$|f(a) - f_n(a)| + V(f - f_n, P) = \lim_{m\to\infty} [| f_n(a) - f_m(a)| + V(f_m - f_n, P)]$$
$$\leq \sup_{m \geq N} \|f_m - f_n\|_{BV} \leq \epsilon.$$

This holds for all P hence

$$\|f - f_n\|_{BV} \leq \epsilon \quad \text{for all } n \geq N.$$

Thus $f_n \to f$ in $\|\cdot\|_{BV}$.

Also, $(f - f_n) \in BV[a,b]$ and $f_n \in BV[a,b]$, hence $f \in BV[a,b]$.

So $BV[a,b]$ with $\|f\|_{BV} = |f(a)| + TV\left(f_{[a,b]}\right)$ is complete.

Notice that convergence in $\|\cdot\|_\infty$ does not imply convergence in $\|\cdot\|_{BV}$. Since

$$f(x) = x\cos\left(\frac{\pi}{2x}\right) \qquad if \quad 0 < x \leq 1$$
$$= 0 \qquad\qquad if \quad x = 0$$

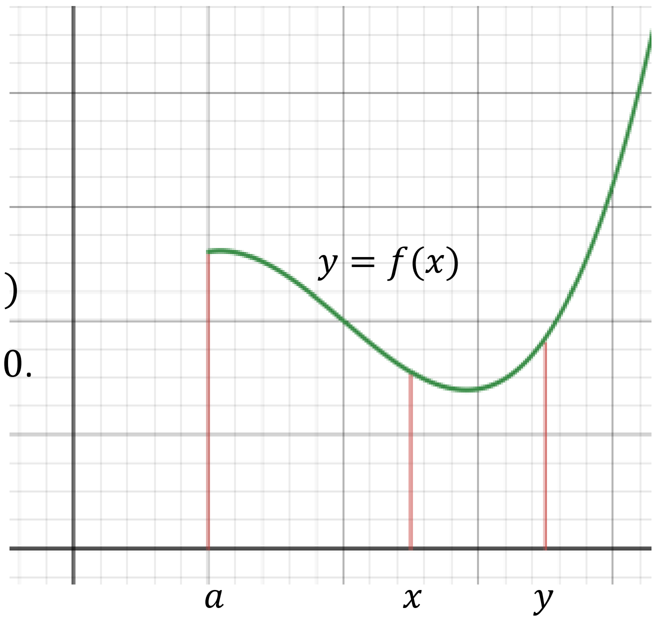
Is continuous on $[0,1]$ we know from the Weierstrass approximation theorem that there is a sequence of polynomials, $p_n(x)$, that converges uniformly to $f(x)$.

But each $p_n(x) \in BV[0,1]$ and $f(x) \notin BV[0,1]$. However, $BV[0,1]$ is complete under $\|\cdot\|_{BV}$ so $\{p_n(x)\}$ can't be a Cauchy sequence in $BV[0,1]$.

Theorem: Fix $f \in BV[a,b]$ and let $v(x) = TV(f_{[a,x]})$, for $a \le x \le b$, and $v(a) = 0$. Then both v and $v - f$ are increasing. Thus

$$f = v - (v - f)$$

Is the difference of two increasing functions.



Proof. For $y > x$ we have:

$$v(y) - v(x) = TV\left(f_{[a,y]}\right) - TV(f_{[a,x]})$$
$$= TV(f_{[x,y]}) \ge |f(y) - f(x)| \ge 0.$$

Hence v is increasing.

But $v(y) - v(x) \ge f(y) - f(x)$ so

$$(v(y) - f(y)) - (v(x) - f(x)) \ge 0.$$

Thus $v - f$ is increasing.

Since monotone functions are of bounded variation we get:

Corollary (Jordan's Theorem) A function $f\colon [a,b] \to \mathbb{R}$ is of bounded variation if and only if f can be written as the difference of two increasing functions.